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Inviscid instability of an incompressible flow between rotating porous cylinders to three-dimensional perturbations

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ABSTRACT

We consider the stability of the Couette–Taylor flow between porous cylinders with radial throughflow in the limit of high radial Reynolds number. It has already been shown earlier that this flow can be unstable to two-dimensional perturbations. In the present paper, we study its stability to general three-dimensional perturbations. In the limit of high radial Reynolds number, we show the following: (i) the purely radial flow is stable (for both possible directions of the flow); (ii) all rotating flows are stable with respect to axisymmetric perturbations; (iii) the instability occurs for both directions of the radial flow provided that the ratio of the azimuthal component of the velocity to the radial one at the cylinder, through which the fluid is pumped in, is sufficiently large; (iv) the most unstable modes are always two-dimensional, i.e. two-dimensional modes become unstable at the smallest ratio of the azimuthal velocity to the radial one; (v) the stability is almost independent of the rotation of the cylinder, through which the fluid is being pumped out. We extend these results to high but finite radial Reynolds numbers by means of an asymptotic expansion of the corresponding eigenvalue problem. Calculations of the first-order corrections show that small viscosity always enhances the flow stability. It is also shown that the asymptotic results give good approximations to the viscous eigenvalues even for moderate values of radial Reynolds number.

1. Introduction

We consider the stability of a steady viscous incompressible flow in a gap between two rotating porous cylinders in the limit of high radial Reynolds number R (constructed using the radial velocity and the radius of the inner cylinder). The basic flow is rotationally and translationally (along the common axis of the cylinders) invariant and generalises the classical Couette–Taylor flow to the case when a radial flow is present. The direction of the radial flow can be from the inner cylinder to the outer one (the diverging flow) or from the outer cylinder to the inner one (the converging flow). It has been shown earlier [1,2] that this flow can be unstable to small two-dimensional perturbations, and the aim of the present paper is to understand what happens if three-dimensional perturbations are allowed.

The stability of viscous flows between permeable rotating cylinders with a radial flow had been studied by many authors [3–9]. One of the main aims of these papers was to determine the effect of the radial flow on the stability of the circular Couette–Taylor flow to axisymmetric perturbations, and the general conclusion was that the radial flow affects the stability of the basic flow: both a converging radial flow and a sufficiently strong diverging flow have a stabilising effect on the Taylor instability, but when a diverging flow is weak, it has a destabilising effect [5,6]. However, it was not clear whether a radial flow itself can induce instability for flows which are stable without it. This question had been answered affirmatively by Fujita et al. [10] and later by Gallet et al. [11] who had demonstrated that particular classes of viscous flows between porous rotating cylinders can be unstable to small two-dimensional perturbations.

Later it had been shown by Ilin and Morgulis [1] that both converging and diverging irrotational flows can be linearly unstable to two-dimensional perturbations in the limit $R \to \infty$ and that the instability persists if small viscosity is taken into account. In Ref. [2], the same limit had been considered for general viscous flows between porous cylinders with a radial flow, and it had been shown that not only the particular classes of viscous steady flows considered in [11,1] can be unstable to

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two-dimensional perturbations, but this is also true without any restriction on angular velocities of the cylinders and for both converging and diverging flows. A further development of the two-dimensional theory can be found in a recent paper by Kerswell [12] where, among other things, the effects of compressibility and nonlinearity have been considered. Kerswell has also pointed to a similarity between the instability induced by the radial flow and the so-called stratorotational instability (SRI) which is due to the axial density stratification of the Couette–Taylor flow (see also [13]).

An interesting and important feature of the basic steady flow considered here is that it strongly depends on the radial Reynolds number R and on the direction of the radial flow. When $R = 0$ (no radial flow), it reduces to the standard Couette–Taylor flow, but when $R \gg 1$, it tends to an inviscid irrotational flow in which both radial and azimuthal components of the velocity are inversely proportional to r (where r is the radial coordinate of the polar cylindrical coordinate system). The parameters of this inviscid flow are determined by the radial and azimuthal components of the velocity at the flow inlet (i.e. at the inner cylinder for the diverging flow or at the outer cylinder for the converging flow) irrespective of what happens at the outlet. This means that a single inviscid flow represents the inviscid limit common for all viscous flows with the same radial mass flux and the same azimuthal velocity at the inlet irrespective of the angular velocity of the other cylinder (which represents the flow outlet). Of course, the inviscid flow approximates the exact viscous flow only outside a thin boundary layer near the flow outlet, and the boundary layer depends on the angular velocity of the other cylinder. However, an asymptotic expansion for $R \gg 1$, constructed in [2] for the two-dimensional problem, shows that the leading term is completely determined by the inviscid stability problem for the basic inviscid flow described above and does not depend on the boundary layer. Interestingly, the first viscous correction term in the expansion also does not depend on the details of the boundary layer in the basic flow, i.e. the first viscous correction does not feel what is happening at the flow outlet.

In the present paper, we examine the effect of three-dimensional perturbations on the stability properties of the basic flow described above for the flow regimes with high radial Reynolds number. We construct an asymptotic expansion of the eigenvalue problem for normal modes for $R \gg 1$, study the inviscid problem in detail and compute the principal viscous corrections to the inviscid eigenvalues. In particular, we rigorously prove that axisymmetric inviscid modes always decay exponentially, as well as all inviscid modes for the purely radial basic flow. The critical curves of the inviscid instability computed numerically show that, for a wide range of the flow parameters and for both diverging and converging flows, the unstable inviscid modes appear as soon as the circulation of the velocity at the flow inlet becomes larger that a certain critical value and that the purely two-dimensional azimuthal waves are always the most unstable ones, i.e. they correspond to the smallest critical value of the inlet circulation. At the same time, the instability is almost independent of the azimuthal velocity at the outlet. This means that the Couette–Taylor flow in the presence of the radial flow can be unstable far beyond the Rayleigh line (that separates inertially stable and unstable regimes in the classical Couette–Taylor flow).

We also calculate viscous corrections and investigate their effect on the instability. In particular, the analysis of the principal viscous corrections shows that, for both the diverging and converging flows, small viscosity always enhances the flow stability.

The outline of the paper is as follows. In Section 2, we discuss the exact viscous basic flow and its inviscid limit and formulate the exact and inviscid linear stability problems. Section 3 contains a linear inviscid stability analysis of both the diverging and converging flows basic flows. In Section 4, the effect of viscosity is considered. Discussion of the results is presented in Section 5.

2. Formulation of the problem

2.1. Exact equations and basic steady flow

We consider three-dimensional viscous incompressible flows in the gap between two concentric circular cylinders with radii r_1 and r_2 ($r_2 > r_1$). The cylinders are permeable for the fluid and there is a constant volume flux $2\pi Q$ (per unit length measured along the common axis of the cylinders) of the fluid through the gap (the fluid is pumped into the gap at the inner cylinder and taken out at the outer one or *vice versa*). Q will be positive if the direction of the flow is from the inner cylinder to the outer one and negative if the flow direction is reversed. Flows with positive and negative Q will be referred to as diverging and converging flows respectively. Suppose that r_1 is taken as a length scale, $r_1^2/|Q|$ as a time scale, $|Q|/r_1$ as a scale for the velocity and $\rho Q^2/r_1^2$ for the pressure where ρ is the fluid density. Then the Navier–Stokes equations, written in non-dimensional variables, have the form

$$u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} = -p_r + \frac{1}{R}\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}v_\theta\right), \tag{1}$$

$$v_t + uv_r + \frac{v}{r}v_\theta + wu_z + \frac{uv}{r} = -\frac{1}{r}\,p_\theta + \frac{1}{R}\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}u_\theta\right), \tag{2}$$

$$w_t + uw_r + \frac{v}{r}w_\theta + ww_z = -p_z + \frac{1}{R}\,\nabla^2 w, \tag{3}$$

$$\frac{1}{r}\,(ru)_r + \frac{1}{r}\,v_\theta + w_z = 0. \tag{4}$$

Here (r, θ, z) are the polar cylindrical coordinates, u, v and w are the radial, azimuthal and axial components of the velocity, p is the pressure, $R = |Q|/\nu$ is the Reynolds number (ν is the kinematic viscosity of the fluid); subscripts denote partial derivatives; ∇^2 is the polar form of the Laplace operator:

$$\nabla^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2.$$

We employ the following boundary conditions

$$u\big|_{r=1} = \beta, \qquad u\big|_{r=a} = \frac{\beta}{a}, \qquad v\big|_{r=1} = \gamma_1, \qquad v\big|_{r=a} = \frac{\gamma_2}{a}, \qquad w\big|_{r=1} = w\big|_{r=a} = 0 \tag{5}$$

where

$$a = \frac{r_2}{r_1}, \qquad \beta = \frac{Q}{|Q|}, \qquad \gamma_1 = \frac{\Omega_1 r_1^2}{|Q|}, \qquad \gamma_2 = \frac{\Omega_2 r_2^2}{|Q|},$$

with Ω_1 and Ω_2 being the angular velocities of the inner and outer cylinders respectively. Parameter β takes values $+1$ or -1 which correspond to the diverging and converging flows respectively; γ_1 and γ_2 represent the ratio of the azimuthal component of the velocity to the radial one at the inner and outer cylinders respectively. Boundary condition (5) prescribe all components of the velocity at the cylinders and model conditions on the interface between a fluid and a porous wall [14].

The only steady rotationally symmetric and translationally invariant (in the z direction) solution of problem (1)–(5) is given by

$$u = \frac{\beta}{r}, \qquad v = V(r) = Ar^{\beta R+1} + \frac{B}{r}, \qquad P = -\frac{1}{2r^2} + \int \frac{V^2(r)}{r}\,dr \tag{6}$$