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Geometrical interpretations of continuous and complex-lamellar steady flows

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ABSTRACT

Three dimensional incompressible steady flows were investigated from a kinematical perspective. It is shown that their geometry, as defined by their streamline curvature, is directly related to the *Curl* of the associated unit tangent vector $\boldsymbol{t}$. This relationship reveals that both the complex-lamellar and the continuity flow conditions impose geometrical constraints. These simplify flow analysis considerably, enabling a pure geometric representation of the underlying physics. Specifically, rather than using the traditional physical variables of vorticity, $\nabla \times \boldsymbol{v}$ and divergence $\nabla \cdot \boldsymbol{v}$, for the flow description, the geometric related variables $\nabla \times \boldsymbol{t}$ and $\nabla \cdot \boldsymbol{t}$, representing the streamline curvature K_S and the mean curvature of the normal congruence H_{nc}, respectively, were considered. The systematic implementation of these mathematical findings leads to the appearance of an interesting equation for the flow velocity, as a function of H_{nc}. Based on this, the concept of "Geometric Lensing" has been introduced. According to this concept, the normal congruence spreads or focuses the flow streamlines through its curvature, thus altering their density and ultimately flow velocity. Geometric Lensing transforms the physical problem of finding the velocity distribution to a purely geometrical one and provides an intuitive explanation of the relieving effect that three dimensional flows exhibit. Finally, the inherent relationship between physical and geometric quantities of irrotational flows is explored. The Geometric Potential Theory, originally developed for planar flows, has been extended in three dimensions. The theoretical findings could provide useful post-processing tools for both experimentalists and CFD engineers, as well as for researchers interested in scientific visualization.

1. Theoretical background and motivation

The rotation of the velocity field $\boldsymbol{v}$, corresponding to fluid flows, is a vector known as vorticity ($\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}$). Its magnitude is twice the mean angular velocity ω that the fluid particles experience ($|\boldsymbol{\Omega}| = 2\omega$) and its direction is along their axis of rotation, as determined by the right-hand rule [1]. Vector fields were initially categorized in to rotational and irrotational fields, based on this kinematical feature of vorticity. It then soon appeared that such classification was also important from a physical point of view. In his paper on the motion of fluids, Euler drew attention to a planar motion of an incompressible fluid that is irrotational at any time [2]. Based on the assumptions that the flow density ρ is constant and that $\boldsymbol{\Omega} = \boldsymbol{0}$, the general equations characterizing the flow,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{v}) = \boldsymbol{0},$$

are reduced to the following two equations:

$$\nabla \cdot \boldsymbol{v} = 0 \quad \textit{Mass conservation for an incompressible fluid (Continuity)}, \tag{1.1}$$

$$\nabla \times \boldsymbol{v} = \boldsymbol{0} \quad \textit{Fluid elements have no angular velocity (Irrotationality)}. \tag{1.2}$$

The fact that the irrotationality condition is not just an assumption of theoretical interest, since it is encountered in nature [3], initiated the development and application of classical potential theory in fluids, a research field that although old is being practiced even today [4,5]. According to this theory, the problem of finding the velocity components is significantly simplified, because these can be determined by a single scalar, the *velocity potential*, Φ such that

$$\boldsymbol{v} = \nabla \Phi. \tag{1.3}$$

Since $\nabla \Phi$ lies in a direction that is perpendicular to a family of surfaces with constant values ($\Phi = \textit{constant}$), irrotationality implies that there is a family of iso-surfaces everywhere normal to the flow streamlines. This geometric characteristic that potential

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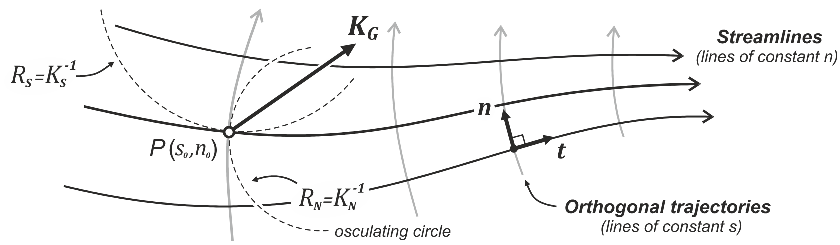

Fig. 1. The Global Curvature vector $\boldsymbol{K_G}$ in the streamline coordinate system (s, n).

function Φ admits, together with the fact that all velocity components can be derived from it (Eq. (1.3)), is evidence that the physics of a potential flow must be contained within its topology. This speculation has been confirmed in the past, for planar flows. Instead of using the "conventional" scalar, namely the velocity potential, to describe their motion, a kinematical interpretation in terms of the local curvatures has been given. This was based on the observation that while the rotation and divergence of the velocity vector have an intrinsic physical content (Eqs. (1.1) and (1.2)), the rotation and divergence of its normalized field ($\boldsymbol{t} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$), suggest a geometric nature. They determine the curvature of the streamlines and their orthogonal trajectories, K_S and K_N, respectively,}[6]:

$$\nabla \cdot \boldsymbol{t} = K_N \quad (\textit{Curvature of the orthogonal trajectories}), \tag{1.4}$$
$$|\nabla \times \boldsymbol{t}| = K_S \quad (\textit{Streamline Curvature}). \tag{1.5}$$

Application of these results to a planar potential flow led to the vector identity [7]:

$$\nabla v = v\boldsymbol{K_G} \quad \text{or equivalently}$$
$$\boldsymbol{K_G} = \nabla N \text{ with } \begin{cases} \boldsymbol{K_G} = (-K_N,\ K_S) \\ N = \ln v,\ v > 0, \end{cases} \tag{1.6}$$

where $\boldsymbol{K_G}$ is the *Global Curvature Vector* and N is the *geometric potential*. The components of $\boldsymbol{K_G}$ are expressed in the streamline coordinate system (s, n), defined by the unit vectors $\boldsymbol{t}$ and $\boldsymbol{n}$, which are tangent to the streamlines and to their orthogonal trajectories, respectively, thus forming an orthogonal curvilinear coordinate system (Fig. 1). In this system, the associated independent variables s and n represent the distance (arclength) along and across the streamlines. In Fig. 1, the dashed lines represent the osculating circles, which are the "best circles"[1] that approximate the curves (streamline and orthogonal trajectory) passing through the point of interest P. The corresponding radius of each circle at a given point is called the radius of curvature of the curve at that point and equals to the reciprocal of the curvature itself. Eq. (1.6) indicates that $\boldsymbol{K_G}$ gives both the direction and magnitude of the maximum spatial percentage rate of increase in the velocity. It follows that its determination can be reduced to the purely geometric problem of finding these two curvatures. This way classical potential theory, expressed through Eq. (1.3), can be replaced by a *Geometric Potential Theory*, or shortly GPT, through Eq. (1.6).

It is reasonable to question whether three dimensional flow fields would also exhibit a geometrical manifestation of their "nature" and if yes, under which circumstances. An appropriate candidate to start with would of course be the class of irrotational flows, which already gave fruitful results in two dimensions. However, it seems that an even broader class of flows, the so-called *Complex-Lamellar* flows, would allow us to mathematically link their topological information to their velocity distributions as well.

[1] The osculating circle is the one among all tangent circles at P that approaches the curve most tightly.

The reason to support this is hidden in the very definition of the complex-lamellar condition, which similar to the irrotationality condition, happens to impose geometrical constraints. More precisely, apart from enforcing the flow to be orthogonal to its vorticity, that is,

$$\boldsymbol{v} \cdot \nabla \times \boldsymbol{v} = 0 \quad \textit{Flow is orthogonal to its vorticity} \quad (\textit{Complex-Lamellar condition}), \tag{1.7}$$

it provides both the necessary and sufficient condition for the existence of a family of surfaces that is orthogonal to the flow streamlines, which is called the *Normal Congruence* of $\boldsymbol{v}$ [8].

The geometric implications associated with the complex-lamellar condition can also be illustrated by following Clebsch's potential approach to fluid dynamics [9]. Accordingly, any arbitrary velocity field $\boldsymbol{v}$ can always be expressed (at least locally) in the following form:

$$\boldsymbol{v} = \nabla\varphi + \lambda\nabla\mu,$$

where φ, λ and μ are scalar functions, also known as Monge's potentials. As a result, the vorticity vector $\boldsymbol{\Omega}$ can be written as

$$\begin{aligned} \boldsymbol{\Omega} &= \nabla \times (\nabla\varphi + \lambda\nabla\mu) \Leftrightarrow \\ \boldsymbol{\Omega} &= \nabla \times \nabla\varphi + \nabla \times (\lambda\nabla\mu) \Leftrightarrow \\ \boldsymbol{\Omega} &= \nabla\lambda \times \nabla\mu, \end{aligned} \tag{1.8}$$

so that the surfaces of constant λ and μ are vortex surfaces and their intersections represent vortex lines. The last equation transforms the complex-lamellar condition to

$$\begin{aligned} &(1.7) \overset{(1.8)}{\Longrightarrow} (\nabla\varphi + \lambda\nabla\mu) \cdot (\nabla\lambda \times \nabla\mu) = 0 \Leftrightarrow \\ &\nabla\varphi \cdot (\nabla\lambda \times \nabla\mu) + \lambda\nabla\mu \cdot (\nabla\lambda \times \nabla\mu) = 0 \Leftrightarrow \\ &\nabla\varphi \cdot (\nabla\lambda \times \nabla\mu) = 0 \Leftrightarrow \\ &\frac{\partial(\varphi, \lambda, \mu)}{\partial(x, y, z)} = 0, \end{aligned} \tag{1.9}$$

in the Cartesian coordinate system. Hence, Eq. (1.9), which is an alternative representation of Eq. (1.7), might perhaps demonstrate that even three dimensional rotational flows could exhibit a special feature of geometric character, so that their physical content is "mirrored" in their appearance.

In order to achieve our goal, it is necessary to first find the means required to express this topological information mathematically. This is achieved in the following section. It is proven that for any three dimensional field $\boldsymbol{v}$, the *Curl* of its unit vector $\boldsymbol{t} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ captures the intrinsic property of streamline curvature entirely. The derived equation for $\nabla \times \boldsymbol{t}$ is eventually analyzed and kinematically interpreted. Consequently, the implications that continuity and the complex-lamellar assumptions have on the flow geometry, have been studied using this equation. Firstly, it is shown that the orthogonality between the unit tangent vector $\boldsymbol{t}$ and its rotation $\nabla \times \boldsymbol{t}$, is sufficient and necessary for a flow to be complex-lamellar (Section 3). Furthermore, the additional possession of the continuity condition enables the establishment of a mathematical