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Shallow waves in density stratified shear currents

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ABSTRACT

In this paper we examine the role of nonlinearity on the evolution of surface and internal layers in density stratified fluids with steady but different shear currents in each stratified layer. As we show, when the difference between the vorticities in each layer is sufficiently large and of different signs, large amplitude nonlinear phenomena, particularly along the internal layer, emerges. Dispersive shock wave and solitary wave phenomena appear in the parameter regimes examined in this work. Our results show that jumps in density and vorticity generate strong nonlinear responses, and therefore fluid models should account for these variations in order to improve their predictive capabilities.

1. Introduction

Multilayer interface models are a common approximation throughout fluid mechanics to examine otherwise intractable problems if dealt with in their full generality [1]. By ignoring viscosity, which would weakly smear any stable interface due to small length scale diffusion, much work has been done in particular on studying nonlinear wave propagation in the presence of density stratification and steady shear currents [2–8]. Density stratification arising from temperature variations [1] supports the propagation of strong nonlinear waves along atmospheric inversion layers and along pycnoclines in the ocean. These waves are significant mechanisms for energy transport due to their long-time stability. Shear currents, arising from wind/wave interactions [4,9] or drag effects along boundaries [4], while not specifically supporting nonlinear waves, have significant impacts on their dynamics.

In the absence of density stratification, the impact of shear currents on surface waves and bulk-flow is a well-studied problem. Starting with the seminal paper of Benjamin [10], which looked at small-amplitude nonlinear traveling waves over arbitrary vertical shear flows, or flows with an arbitrary specification of the vorticity of the flow, research has shown that vorticity strongly modifies the shape of solitary waves leading in some cases to overturning waves [4,11–14], inviscid eddy-formation [4,15], and non-monotone pressure distributions [4,16,17]. A large body of rigorous results concerning the existence and properties of steady periodic waves over steady shear currents has appeared as well. See [18] and its extensive bibliography for reference. We also note that [19] numerically examines the impact of bilinear shear profiles, in which the vorticity has a discontinuous jump between two constant values; see Figs. 1 and 2 for reference. Nonsteady wave propagation has been looked at first in [20], experimentally and numerically in [21], and recently in [22], which looks at fully three-dimensional profiles. We make particular note of [20], which was the first paper to examine the case of a bilinear shear flow. Recent results in [23,24] have developed free surface models with arbitrary vorticity profiles but with constant density throughout the fluid.

Density stratified shear flows have likewise received much attention over the years. In parallel with the description of bilinear shear currents, we refer to fluid systems composed of two fluids of differing constant densities, as bistratified fluids. Nonlinear, large amplitude steady waves for bistratified fluids were studied in [6,8], while in [25], the unsteady case is examined. As shown throughout these papers, strong internal waves are found, and a complete characterization of the linear stability of small amplitude internal waves in the presence of a bilinear shear current was presented in [7]. In more general circumstances, the KdV and Benjamin–Ono equations were derived as models of long waves in the presence of arbitrary vertical shear and density variations [26]. As noted in [26], given the interest in and importance of internal solitary waves in oceanic flows [27–30], an understanding of the influence of shear currents is necessary to better model flow through channels, estuaries, or relatively shallow water environments [31].

However, previous works on stratified shear flow have usually focused on internal layers alone by either assuming rigid lids both

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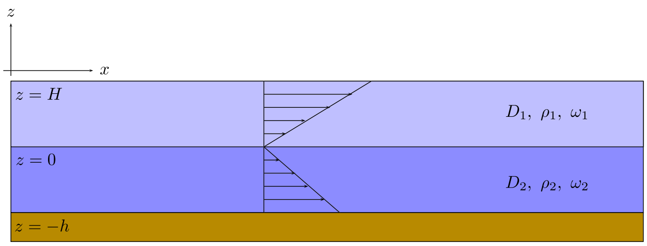

Fig. 1. A discontinuous bilinear shear profile characterized by constant vorticities ω_j in a bistratified fluid with densities ρ_j. In the figure, $\omega_1 > 0$ and $\omega_2 < 0$. Each stratified layer is referred to by D_j. Note the background velocity is continuous through the line $z = 0$.

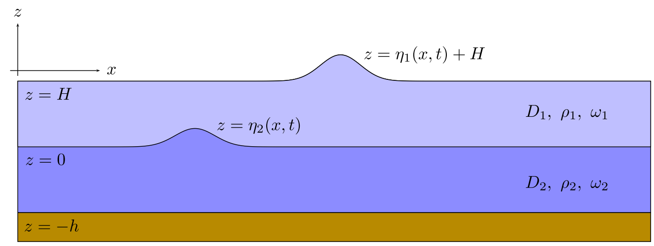

Fig. 2. Free surfaces propagating over the surface, i.e. $z = \eta_1(x, t) + H$, layer and internal, i.e. $z = \eta_2(x, t)$, layer. The resting fluid depths are $z = H$, $z = 0$, and the bottom boundary is at a depth $z = -h$.

above and below the fluid [6–8,25,32], or by taking boundaries to infinity [26]. Therefore, in this paper we look at the evolution of a free surface and a free internal layer for a bilinear shear current in a bistratified fluid; see Fig. 2 for reference.

Using the techniques in [33,34], a closed system describing the evolution of the two free surfaces and their corresponding velocity potentials are derived. We also do not assume the two layers are of equal depth, which introduces the depth ratio parameter $\bar{d}$. Using this new formulation, we investigate two asymptotic regimes; a small amplitude regime, and a shallow-water long-wavelength regime. Using a small amplitude approximation (*i.e.*, linearizing about the zero solution) we study whether Kelvin–Helmholtz (KH) instabilities form along either interface. As we show, for $\bar{d}$ sufficiently small, or if the upper layer is sufficiently deeper than the lower layer, and $\rho = \rho_2/\rho_1 \gg \bar{d} + 1$, there is no KH instability for systems in which the upper layer has relatively weak vorticity. We provide numerical evidence that this result extends beyond the assumptions that $\bar{d} \ll 1$ and $\rho \gg 1 + \bar{d}$. We emphasize that we have not included surface tension and that the background velocity profile is continuous through the flat interface, and so at least at the linearized level, differences in background vorticity can suppress the KH instability for certain parameter regimes. However, we also point out that KH instabilities are also found for some parameter choices. Our results qualitatively agree with some aspects of the work in [35]. For example, as we decrease the density ratio from $\rho = 820$ to $\rho = 1.5$, we see stability regions shrink. However, vorticity is not present in the bulk of the fluid in the models studied in [35], and as such, those results also rely on the jump in the interface velocity and the stabilizing role of surface tension.

To characterize the impact of nonlinearity on bistratified, bilinear shear systems, using a shallow water, long wavelength ansatz, the flow is separated into four separate waves characterized by four different wave speeds and four uncoupled Korteweg–de Vries (KdV) equations describing the long time evolution of each separated wave. While exact formulas for the wave speeds and KdV coefficients are not given due to their complexity, they are readily computed numerically, and characterizations of each value as functions of the stratification ρ and the vorticities ω_j are given. By making a particular assumption for the initial surface and internal wave heights and tangential velocities, and choosing $\rho = 820$, which is the average ratio in density between air and water [36], and $\bar{d} = 0.25$, we show that for relatively large differences in the vorticities ω_j that strong nonlinear waves along the interface form and propagate. While the constant vorticity case, i.e. $\omega_1 = \omega_2$, can lead to the formation of strong nonlinear waves along interfaces, the greater difference in shear profiles results in significantly stronger nonlinear responses. Likewise, the formation of dispersive shock waves in our simulations shows that only looking for traveling wave solutions does not capture all of the interesting physics that one might observe. To complement these results, we look at the much different case of nearly equal densities by choosing $\rho = 1.5$ and $\bar{d} = 4$. Again, large vorticity differences lead to the formation of stronger nonlinear waves than just the $\omega_1 = \omega_2$ case would predict, though the difference is not as drastic as in the case of $\rho = 820$. Again, the results found in [35] are similar in spirit, though the presence of vorticity in each fluid layer makes our work allows for some significant differences to the results in [35].

So while these results show that significant nonlinear effects can appear in bistratified, bilinear shear currents, they also hint at more interesting effects that might appear in the presence of more shear layers. We also do not address the case of the density stratification ratio ρ being very close to unity. Given that the average stratification ratio in the ocean is $\rho = 1.028$, this is certainly a problem worthy of study. However, our approach would necessitate a higher order expansion in order to capture this new scale. Finally, we do not address the case of varying bathymetry. Modeling multiple layers, very weak stratification, and varying bathymetry are topics of future research. Potential applications of our results could be to the emerging field of wave-energy extraction where air and underwater currents should be taken into account during modeling in order to generate accurate approximations of sea states. However, we note that our results do not take into account thermal fluctuations, viscosity, or the Earth's rotation, though for near shore applications these inadequacies of our model may not be significant. See for example [37] which studies wave-energy extraction in the incompressible, irrotational, and inviscid regime.

In the following section, we present the details of the formulation of the density stratified, bilinear shear current system. In Section 3, we present the derivation of the surface variable formulation of the problem described in Section 2. The presence of KH instabilities is discussed in this section. In Section 4, we find the shallow water, long-wavelength approximation which leads to the derivation of four de-coupled KdV equations. We also look briefly at the linear dynamics of this system. In Section 5, we study the evolution of the KdV equations and the nonlinear phenomena that results due to the varying shear. Section 6 concludes the paper and discusses future directions, while the Appendix collects techincal details.

2. Problem formulation

We now examine the case of unsteady nonlinear wave propagation over a bistratified, bilinear shear current as illustrated in Fig. 1. Throughout, we let D_1 represent the top fluid domain, and D_2 represent the bottom fluid domain. In each layer, we assume the fluid to be both inviscid and incompressible with the only external force being that of gravity. The interfaces are assumed to be free surfaces described by $z = \eta_1(x, t) + H$ and $z = \eta_2(x, t)$. Furthermore, we suppose that there is a flat bottom at $z = -h$ through which no fluid flows. See Fig. 2 for reference.

Restricting ourselves to a linear shear, or constant vorticity, background flow within each layer of the fluid ($j = 1, 2$), Euler's equations of motion for $(x, z) \in D_j$ become

$$\nabla \cdot \mathbf{u}_j = 0, \tag{1}$$

$$\nabla \times \mathbf{u}_j = \omega_j \hat{\mathbf{y}} \tag{2}$$

$$\partial_t \mathbf{u}_j + \mathbf{u}_j \cdot \nabla \mathbf{u}_j = -\frac{1}{\rho_j} \nabla p_j - g\hat{\mathbf{z}}, \tag{3}$$

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