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Flow of a class of fluids defined via implicit constitutive equation down an inclined plane: Analysis of the quasi-steady regime

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HIGHLIGHTS

- We consider a rate-type fluid defined by an implicit constitutive equation.
- We formulate a mathematical model for the flow down an inclined plane.
- We consider lubrication approximation.
- We find analytical solutions.
- We perform numerical simulations.

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ABSTRACT

We consider the motion of a rate-type fluid defined by an implicit constitutive equation down an inclined plane. We assume that the characteristic height of the layer is small in comparison to the characteristic length, so that lubrication approximation can be applied. After re-scaling the governing equations we focus on the leading order approximation and we consider the quasi-steady regime which occurs when the velocity of the advancing front and the velocity of the fluid at the inlet are substantially different. We write the differential equation for the evolution of the upper free surface and solve it numerically, plotting the profile of the layer together with the evolution of the advancing front. A comparison with the Newtonian model is also presented, with particular emphasis on the motion of the front.

1. Introduction

In a series of papers [1–3] Rajagopal et al. discussed a class of implicit constitutive models that generalizes the classical Stokesian models in which the stress is explicitly expressed in terms of the kinematical quantities (e.g. Newtonian fluids).

In the classical approach, Stokesian fluids are indeed described by constitutive relations where the Cauchy stress[1] $\mathbb{T}^*$ is expressed in terms of the symmetric part of the velocity gradient $\mathbb{D}^*$, so that $\mathbb{T}^* = -p^*\mathbb{I} + \mathfrak{f}^*(\mathbb{D}^*)$, where p^* is the Lagrange multiplier due to incompressibility and $\mathfrak{f}^*$ is a tensorial function. Under the constraint $\text{tr}\mathbb{D}^* = 0$ (incompressibility), the extra stress $\mathbb{S}^* = \mathbb{T}^* - 1/3(\text{tr}\mathbb{T}^*)\mathbb{I}$ is traceless, so that $p^* = -(1/3)\text{tr}\mathbb{T}^*$ and

$$\mathbb{S}^* = \mathfrak{f}^*(\mathbb{D}^*). \tag{1}$$

Eq. (1) is a special subclass of the more general implicit relation

$$\mathfrak{h}^*(\mathbb{T}^*, \mathbb{D}^*) = 0. \tag{2}$$

When $\mathfrak{f}^*$ is invertible, relation (1) can be rewritten as[2]

$$\mathbb{D}^* = \mathfrak{f}^{*-1}\left(\mathbb{S}^*\right) = \mathfrak{f}^{*-1}\left(\mathbb{T}^* - \frac{1}{3}(\text{tr}\mathbb{T}^*)\mathbb{I}\right) = \mathfrak{g}^*(\mathbb{T}^*). \tag{3}$$

It is important to observe that, in general, models of type (2) cannot be transformed into models of type (3). In [4], for instance, Malek et al. investigated a particular model of type (3), namely

$$2\mu^*\mathbb{D}^* = \left[\alpha\left(1 + \beta^* II_{\mathbb{S}}^{*2}\right)^n + \gamma\right]\mathbb{S}^*, \quad \text{with } II_{\mathbb{S}}^{*2} = \frac{1}{2}\mathbb{S}\cdot\mathbb{S}, \tag{4}$$

where α, β^*, γ are positive constants, n is a real number and μ^* represents fluid viscosity.

The advantage of using generalized models such as (2) or (3) becomes evident when we consider fluids with pressure

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[1] The symbol "*" denotes a dimensional quantity.

[2] Usually, fluids of type (3) are called "stress power-law fluids".

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dependent viscosity as the one studied in [5–8], or when we have to fit the experimental data as in [9,10]. In the latter case, for instance, the reported shear stress versus shear rate graphs have the characteristic S-shape and the experimental data cannot be fitted using constitutive relations of type $\mathbb{S}^* = \mathfrak{f}^*(\mathbb{D}^*)$.

Another interesting example in which the constitutive model cannot be derived within the classical framework (1) is the Bingham visco-plastic model or its generalization (see, for instance, [11–14] where the Bingham model is derived within the implicit constitutive framework). The Bingham fluid is characterized by a critical stress threshold τ_o^* that must be overcome in order to start the flow. Whenever the shear stress is below τ_o^*, the fluid behaves as a rigid body, i.e. $\mathbb{D}^* = 0$. As a consequence, the stress is not a single-valued function of the strain rate, i.e. the relation between stress and velocity gradient cannot be written as $\mathbb{S}^* = \mathfrak{f}^*(\mathbb{D}^*)$. On the contrary, if we use a relation as (2), namely

$$\mathbb{D}^* - \left(\frac{II_{\mathbb{D}^*}}{2\mu^* II_{\mathbb{D}^*} + \tau_o^*}\right)\mathbb{S}^* = 0,$$

the Bingham model produces no indeterminacy. Following [15], in this paper we extend the model (4) to include a stress relaxation term. In practice we consider a "viscoelastic implicit fluid" whose constitutive equation is

$$\lambda^* \overset{\triangledown}{\mathbb{S}^*} + \left[\alpha\left(1+\beta^* II_{\mathbb{S}}^{*^2}\right)^n + \gamma\right]\mathbb{S}^* = 2\mu^*\mathbb{D}^*, \tag{5}$$

where $\triangledown$ stands for the upper convected time derivative (also known as Oldroyd time derivative)

$$\overset{\triangledown}{\mathbb{S}^*} = \frac{D\mathbb{S}^*}{Dt^*} - \mathbb{L}^*\mathbb{S}^* - \mathbb{S}^*\mathbb{L}^{*^T},$$

where $\mathbb{L}^*$ is the gradient of the velocity field. We analyze the thin-layer flow of a fluid of type (5) down an inclined plane (as shown in Fig. 1), when the only driving force is gravity.

This particular problem has several geophysical applications such as lava flows [16,17] and avalanches [18]. Of course other rheologies can be considered, such as the Newtonian flow model [19], the viscoplastic model [20,21], the viscoelastic model [22], the power-law model [23,24] or the one in which viscosity depends on both the pressure and the shear rate presented by Rajagopal et al. [25] and by Saccomandi and Vergori [26].

Here, starting from constitutive equation (5), we consider four different models obtained selecting different values of the parameters $\lambda^*, \alpha, \beta^*$ and γ.

We study the downhill flow assuming that the tilt angle of the inclined plane is "small" and that the characteristic height of the fluid layer is small compared to the characteristic length (lubrication regime). The upper surface of the fluid and the advancing front are unknown and have to be determined imposing a kinematical condition. We focus on the so-called quasi-steady approximation, assuming that the system evolves through steady states. In section Section 3 we show that such a simplification is acceptable from a physical point of view if an appropriate flow regime is considered.

The paper develops as follows: in Section 2 we formulate the mathematical model. Section 3 is devoted to the flow model and non-dimensionalization. In Section 3.1, we illustrate the assumptions concerning the order of magnitude of the characteristic numbers appearing in the model, deriving the quasi-steady approximation. The analysis of the mathematical problem begins in Section 4, where the simple Newtonian model is considered. In Remark 1 we investigate also the case in which, in place of the inlet discharge, the inlet thickness of the layer is prescribed. In Section 5 we consider two examples in which the relaxation term vanishes, while Section 6 is devoted to the visco-elastic implicit model. For each case we write the differential equation for the evolution of the free surface and we solve it numerically, plotting the evolving profile of the fluid layer and the advancing front.

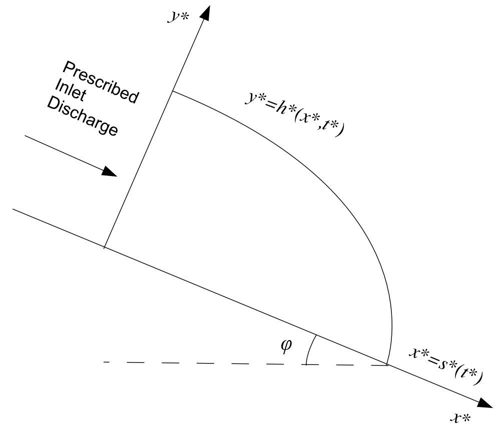

Fig. 1. Flow down an inclined plane. The advancing front is $s^*(t^*)$, whose characteristic velocity, denoted as $\dot{s}_c^*$, is given by (12).

2. The mathematical model

Let us consider a two-dimensional flow down an inclined plane as the one depicted in Fig. 1, and suppose that the only driving force is gravity. The velocity field has the form

$$\boldsymbol{v}^* = u^*(x^*, y^*, t^*)\boldsymbol{e}_x + v^*(x^*, y^*, t^*)\boldsymbol{e}_y.$$

The upper free surface $y^* = h^*(x^*, t^*)$ must fulfill the kinematical conditions

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial}{\partial x^*}\left[\int_0^{h^*} u^*(x^*, y^*, t^*)dy^*\right] = 0, \tag{6}$$

expressing the fact that $h^*(x^*, t^*)$ is a material surface.

The unknown advancing front $x^* = s^*(t^*)$ evolves according to the following balance equation (see formula (2.10) of [19])

$$Q^*\left(t^*\right) = \int_0^{s^*} h^*(x^*, t^*)dx^*, \tag{7}$$

where

$$Q^*\left(t^*\right) = \int_0^{t^*} q^*(\tau^*)d\tau^*,$$

and where $q^*(t^*)$ is the discharge at the inlet $x^* = 0$. In this paper we assume that $q^*(t^*)$ is given, i.e. there is an infinite reservoir from which the fluid is continuously supplied at $x^* = 0$ in a prescribed way.

We denote by φ the tilt angle, by L^* the longitudinal length scale and by H^* the characteristic thickness of the fluid layer. Next, we assume that

$$\varepsilon = \frac{H^*}{L^*} \ll 1,$$

so that lubrication approximation can be applied.

The constitutive model (5) can be rewritten as

$$\lambda^* \overset{\triangledown}{\mathbb{S}^*} + \mathcal{F}^*(II_{\mathbb{S}}^{*^2})\mathbb{S}^* = 2\mu^*\mathbb{D}^*, \tag{8}$$

where $\mathcal{F}^*$ is given by

$$\mathcal{F}^*(II_{\mathbb{S}}^{*^2}) = \left[\alpha(1+\beta^* II_{\mathbb{S}}^{*^2})^n + \gamma\right], \quad \text{with } II_{\mathbf{S}}^{*^2} = \frac{1}{2}\mathbb{S}\cdot\mathbb{S}. \tag{9}$$

In particular, α, γ are dimensionless while $[\beta^*] = \textit{pressure}^{-2}$ and $[\mu^*] = \textit{pressure} \cdot \textit{time}$. When $\alpha = 0, \gamma = 1$, we recover the classical upper convected Maxwell fluid model. Furthermore, if $\lambda^* = 0$, we recover the classical Newtonian model.