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An analytical solution for probability density function of stretching rate in homogeneous isotropic turbulence

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ABSTRACT

A closed-form expression is derived for the probability density function (pdf) of the stretching rate in homogeneous isotropic turbulence. The pdf has the Reynolds-number-independent Gaussian part and Reynolds-number-dependent non-Gaussian part. The physical meanings of the relevant parameters are discussed, and particularly the Reynolds-number effects on the pdf and statistical quantities are examined.

1. Introduction

The stretching rate (the SR in short) is the main source of random stretching of the vorticity field in the production of the enstrophy or the dissipation rate in homogeneous isotropic turbulence. The SR plays an important role in the mathematical theory of the Euler and Navier–Stokes equations [1], which is directly responsible to the formation of singularities and intermittency of incompressible turbulent flows [2–4]. In addition, the SR is directly related to the evolution of material elements in turbulent mixing and diffusion. In this sense, the understanding of the SR statistics is naturally the first step towards the statistics of the dissipation rate, providing a necessary building block in further development based on classical theoretical models of homogeneous isotropic turbulence [5–7]. The statistical data of the SR were obtained mainly from direct numerical simulations (DNS) based on the Navier–Stokes (NS) equations. The probability density functions (pdfs) of the SR calculated in DNS indicate a near-Gaussian distribution with a deviation in the tails [8–11]. In addition, three-dimensional (3D) velocity measurements indicate that although the pdf of the SR could be near-Gaussian in the central region, the significant deviation from a Gaussian distribution toward the negative side was observed in the tails [12–15].

To shed some insights into the statistics of the velocity derivatives, it is tempting to adapt stochastic differential equation models in statistical mechanics [16–18]. Since the non-Gaussian behaviors of the relevant quantities prevail in turbulence, various phenomenological models were proposed as the alternatives of the Kolmogorov's original model (K41) and refined model (K62) [19–23]. Nevertheless, the connection of these non-Gaussian models to the NS equations is tenuous, and there is no theoretical justification for the assumed pdfs [24]. An attempt was made to compute the pdf of the spatial velocity gradients from an evolution equation for the mapping function [25,26]. To understand the non-Gaussian properties of the velocity gradient, this mapping method tried to incorporate certain dynamical aspects of the pdf by mimicking the transport process of turbulence. However, the evolution equation for the mapping function is not directly derived from the NS equations. The non-Gaussian statistics of the velocity increments was studied based on the evolution equation derived from the equation for the velocity gradient tensor [27,28], indicating that the pdf of the longitudinal velocity increment could be significantly deviated and skewed from a Gaussian distribution.

The objective of this work is to obtain a closed-form solution for the pdf of the SR. First, a transport equation for the SR is derived from the NS equations, in which the terms explicitly related to the SR are extracted. Then, under some reasonable assumptions and approximations for homogeneous isotropic turbulence, this transport equation is recast into a random ordinary differential equation, and further an asymptotic analytical solution of the Fokker–Planck equation is obtained for the pdf of the SR fluctuation, which gives a distribution that can be factored into the Gaussian, exponential and higher-order exponential parts. The physical meanings of the relevant parameters are discussed, and

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particularly the Reynolds-number effects on these parameters and the statistical quantities are evaluated. The behavior of the non-Gaussian part is explored by examining the associated potential surface in the phase space.

2. Transport equation for stretching rate

The stretching rate (the SR) of a fluid velocity field is defined as

$$S = e_i e_j u_{j,i}, \quad (i, j = 1,\ 2,\ 3) \tag{1}$$

where e_i are the components of a unit random directional vector, u_i are the fluid velocity components, and $u_{j,\,i} = \partial u_j/\partial x_i$ is the velocity gradient tensor. Here, the index summation is applied. The dyadic product $e_i e_j$ represents the directional projection on the unit vector when it is applied to a second-order tensor. When e_i is defined as the unit separation vector $e_i = r_i/r$ between two fluid particles in turbulence, we have $S = r_i r_j u_{j,i}\, r^{-2}$ that plays an important role in inertial particle clustering in turbulence [29]. Similarly, $S = l_i l_j u_{j,i}\, l^{-2}$ for stretching of a material line is the fundamental quantity in turbulent mixing and diffusion, where l_i/l is the unit directional vector of a material line [8]. Stretching of a vorticity line characterized by $S = \omega_i \omega_j u_{j,i} \omega^{-2}$ plays a critical role in the evolution equation of the enstrophy $\omega^2 = \omega_i \omega_i$, i.e., $d\omega^2/dt = 2S\,\omega^2$ when the viscosity is neglected. Therefore, the SR is the key physical quantity for understanding of the statistics of the enstrophy and dissipation rate in the energy cascade in turbulence. Although the unit directional vector e_i in Eq. (1) can be defined differently depending on a specific application, a generic transport equation for S could be derived.

From the NS equations for an incompressible flow

$$u_{i,\,t} + u_j u_{i,\,j} = -\rho^{-1} p_{,\,i} + \nu\, u_{i,jj}, \tag{2}$$

we have

$$\begin{aligned}\frac{DS}{Dt} &= u_{i,\,k}\frac{D\left(e_i e_k\right)}{Dt} - \rho^{-1} e_i e_k p_{,\,ik} - e_i e_k u_{j,\,k} u_{i,\,j} \\ &\quad + \nu\left[S_{,\,jj} - \left(e_i e_k\right)_{,\,jj} u_{i,\,k} - 2\left(e_i e_k\right)_{,\,j} u_{i,\,jk}\right],\end{aligned} \tag{3}$$

where $e_i e_k$ is a dyadic product of the unit random vectors, u_i are the fluid velocity components, $D/Dt = \partial/\partial t + u_j\partial/\partial x_j$ is the material derivative, p is the fluid pressure, $u_{i,\,t}$ denotes $\partial u_i/\partial t$, and $u_{i,jj}$ denotes $\partial^2 u_i/\partial x_j \partial x_j$. The dyadic product $e_i e_k$ represents the projection on the unit directional vector when it is applied to a second-order tensor.

In order to extract some terms explicitly related to S in the right-hand side (RHS) of Eq. (3), the decompositions of the second-order tensors into an isotropic part and an anisotropic part are used. The velocity gradient tensor $u_{i,\,k}$ is decomposed into

$$u_{i,\,k} = \varGamma\,\delta_{ik} + \varOmega_{ik}, \tag{4}$$

where δ_{ik} is the Kronecker delta, $\varGamma$ is a random isotropic velocity gradient magnitude, and $\varOmega_{ik}$ is the remaining anisotropic tensor. For $i = k$, due to the incompressibility condition $u_{i,\,i} = 0$, a constrain is $\varOmega_{ii} = -3\varGamma$. The SR can be expressed as $S = e_i e_j\, u_{j,\,i} = \varGamma + \varOmega$, where $\varOmega = e_i e_j\, \varOmega_{ij}$. In fact, the rationale behind such tensor decomposition is the Cayley–Hamilton theorem [28,30]. According to the Cayley–Hamilton theorem, we have $u_{i,\,m}u_{m,\,k}u_{k,\,j} + P_s U_{\ i,k}u_{k,j} + Q_S u_{i,j} + R_S\delta_{ij} = 0$, where the first, second and third invariants are $P_S = -u_{i,\,i}$, $Q_S = -u_{i,\,m}u_{m,\,i}/2$, and $R_S = -u_{i,\,m}u_{m,\,n}u_{n,\,i}/3$, respectively. For an incompressible flow with $P_S = -u_{i,\,i} = 0$, $\varGamma$ and $\varOmega_{ik}$ are related to the invariants of $u_{i,\,k}$, i.e., $\varGamma = -R_S Q_S^{-1}$ and $\varOmega_{ik} = -Q_S^{-1} u_{\ i,\,m} u_{m,\,n}\, u_{n,\,k}$.

Further, to evaluate the pressure term and other terms, the decomposition for the directional tensor $e_i e_k$ is formally applied, i.e.,

$$e_i e_k = E\,\delta_{ik} + F_{ik}, \tag{5}$$

where E is a random isotropic magnitude of $e_i e_k$, and F_{ik} is the remaining anisotropic tensor. For $i = k$, the constraints are $e_i e_i = 1$ and is $E = (1 - F_{ii})/3$. The tensor decomposition based on the Cayley–Hamilton theorem leads to the following relations $E = -R_E/Q_E$ and $F_{ik} = -P_E Q_E^{-1} e_i e_m e_m e_k - Q_E^{-1} e_i e_m e_m e_n e_n e_k$, where the first, second and third invariants are $P_E = -e_i e_i = -1$, $Q_E = -e_i e_m e_m e_i/2$, and $R_E = -e_i e_m e_m e_n e_n e_i/3$. The pressure term is written as $\rho^{-1} e_i e_k p_{,\,ik} = \rho^{-1} E\, p_{,\,ii} + \rho^{-1} F_{ik}\, p_{,\,ik}$, where $\rho^{-1} p_{,\,ii} = -u_{i,\,j}u_{j,\,i} = 3\varGamma^2 - \varOmega_{ij}\varOmega_{ji}$ [31].

Eq. (4) is substituted into Eq. (3), which leads to an intermediate equation that contains a quadratic term of S. Further, by using Eq. (5), other terms particularly the pressure term are evaluated, and the additional linear and quadratic terms of S are extracted. After some algebra, we have obtain the following transport equation for S

$$\frac{DS}{Dt} = f - \left(a\,S + b\,S^2\right) + \nu\, S_{,jj} + 6\nu\, E_{,j}\left(SF_{ii}^{-1}\right)_{,j}, \tag{6}$$

where the coefficients are defined as

$$a = 3F_{ii}^{-1}\left(\frac{DE}{Dt} - \nu\, E_{,\,jj}\right) + 2EF_{jk}\varOmega_{jk}F_{ii}^{-2}\left(4E^2 - 3 + 3F_{ii}\right), \tag{7a}$$

$$b = 1 + 3EF_{ii}^{-2}\left(1 - 3E\right), \tag{7b}$$

$$\begin{aligned} f \ =\ & 3F_{jk}\varOmega_{jk}F_{ii}^{-1}\left(\frac{DE}{Dt} - \nu\, E_{,\,jj}\right) \\ & + \varOmega_{ik}\left(\frac{DF_{ik}}{Dt} - \nu\, F_{ik,\,jj}\right) - \rho^{-1} F_{ik} p_{,\,ik} \\ & + \left[\left(9 - 3E\right)F_{ii}^{-2} + 6EF_{ii}^{-1} + 1\right]\left(F_{jk}\varOmega_{jk}\right)^2 - F_{ik}\varOmega_{jk}\varOmega_{ij} \\ & - \nu\left[6E_{,\,j}\left(F_{jk}\varOmega_{jk}F_{ii}^{-1}\right)_{,\,j} + 2F_{ik,\,j}\varOmega_{ik,\,j}\right]. \end{aligned} \tag{7c}$$

The physical meanings of the relevant terms in Eq. (6) are discussed. The transport of S in turbulence is driven by the random force f that collectively incorporates all complex interactions between the anisotropic parts of the velocity derivative tensor and the directional tensor and the projected second-order derivative tensor of pressure projected onto the anisotropic part of the directional tensor. The term $a\,S + b\,S^2$ in Eq. (6) is a damping term. The term $\nu\, S_{,jj}$ is a homogeneous viscous diffusion, while $6\nu\, E_{,j}\left(SF_{ii}^{-1}\right)_{,j}$ is interpreted as a non-homogeneous viscous flux of S projected along the direction of the gradient $E_{,j}$. The factor $F_{jk}\varOmega_{jk}$ in a and f is interpreted as the projection of the anisotropic part of the velocity derivative tensor $u_{j,\,i}$ onto the anisotropic part of the directional sensor $e_i e_k$. This term represents interactions between the anisotropic parts of the velocity derivative tensor and the directional sensor. Similarly, $F_{ik}\varOmega_{jk}\varOmega_{ij}$ represents the triple interaction. The term $F_{ik,\,j}\varOmega_{ik,\,j}$ represents interaction between the derivatives of the anisotropic parts of the velocity derivative tensor and the directional tensor. Formally, $DE/Dt - \nu\, E_{,\,jj}$ and $DF_{ik}/Dt - \nu\, F_{ik,\,jj}$ in Eqs. (7a) and (7c) represent the virtual source terms in the transport of E and F_{ik}, respectively. The term $F_{ik}p_{,\,ik}$ is the projection of the second-order derivative tensor of pressure (the pressure Hessian) onto the anisotropic part of the directional sensor $e_i e_k$. In fact, the pressure Hessian can be similarly decomposed into the isotropic local term and the anisotropic term that represents the non-local contribution [10].

3. Random differential equation for stretching rate

Homogeneous isotropic turbulence is considered, where the statistical properties of the random variables are independent of the position. In a frame moving with a fluid element, since $D/Dt = d/dt$, the quantities in Eq. (6) are considered as a function of time only. Furthermore, in the inertial range, the viscous terms can be