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# An inverse problem of finding the time-dependent thermal conductivity from boundary data

M.J. Huntul[a,b,*], D. Lesnic[a]

[a] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
[b] Department of Mathematics, Faculty of Science, Jazan University, P.O. Box 114, Jazan 45142, Saudi Arabia

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ABSTRACT

We consider the inverse problem of determining the time-dependent thermal conductivity and the transient temperature satisfying the heat equation with initial data, Dirichlet boundary conditions, and the heat flux as overdetermination condition. This formulation ensures that the inverse problem has a unique solution. However, the problem is still ill-posed since small errors in the input data cause large errors in the output solution. The finite difference method is employed as a direct solver for the inverse problem. The inverse problem is recast as a nonlinear least-squares minimization subject to physical positivity bound on the unknown thermal conductivity. Numerically, this is effectively solved using the *lsqnonlin* routine from the MATLAB toolbox. We investigate the accuracy and stability of results on a few test numerical examples.

## 1. Introduction

In inverse problems, the unknown densities or distributed source, or the coefficients involved in the governing partial differential equation or in the boundary conditions for a mathematical model under investigation are sought from additional information on the main dependent variable solution of the original direct initial boundary value problem, [11]. In particular, the inverse problem of identifying the thermal diffusivity/conductivity from boundary data (temperature and partial heat flux) has been investigated widely by many researchers in the past, see [1-3,5-8,12] to mention only a few. In this paper, the novelty consists in the development of a convergent numerical optimization method for solving this nonlinear but well-posed inverse coefficient problem for the heat equation. Numerically, the implementation is realised using the MATLAB toolbox routine *lsqnonlin*.

The paper is organized as follows. In Section 2, the mathematical formulation of the inverse problem is presented. In Section 3, the numerical solution of the direct problem is based on the finite difference method with the Crank-Nicolson scheme. In Section 4, the minimization algorithm to solve the inverse problem is presented. The numerical results are discussed in Section 5. Finally, conclusions are highlighted in Section 6.

## 2. Mathematical formulation

In the domain $Q_T = \{(x,t)|0 < t < T,\ 0 < x < L\}$, we consider the inverse problem given by the parabolic heat equation

$$\frac{\partial u}{\partial t}\left(x,t\right) = a(t)\frac{\partial^2 u}{\partial x^2}\left(x,t\right) + f\left(x,t\right), \qquad \left(x,t\right) \in Q_T, \tag{1}$$

with known heat source $f(x,t)$, unknown temperature $u(x,t)$ and unknown thermal conductivity $a(t) > 0$, $t \in (0,T]$, subject to the initial condition

$$u(x,0) = \phi(x), \qquad x \in [0,L], \tag{2}$$

the Dirichlet temperature boundary conditions

$$u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t), \qquad t \in [0,T], \tag{3}$$

and the Neumann heat flux overdetermination condition

$$a(t)u_x(0,t) = \mu_3(t), \qquad t \in [0,T]. \tag{4}$$

The uniqueness of solution of the inverse problems (1)–(4) has been established in [6] and reads as follows.

**Theorem 1. (Uniqueness of the solution).** *If* $0 < \mu_3 \in C[0,T]$, *then a solution* $(a(t), u(x,t)) \in H^{1+\alpha/2}[0,T] \times H^{2+\alpha,1+\alpha/2}(\overline{Q_T})$ , *for some* $\alpha \in (0,1)$, $a(t) > 0$ *for* $t \in [0,T]$, *to the problem* (1)–(4) *is unique.*

In this theorem, $H^{1+\alpha/2}[0,T]$ denotes the space of Hölder continuously differentiable functions on $[0,T]$ with exponent $\alpha/2$. Also, $H^{2+\alpha,1+\alpha/2}(\overline{Q_T})$ denotes the space of continuous functions $u$ along with their partial derivatives $u_x$, $u_{xx}$, $u_t$ in $\overline{Q_T}$, with $u_{xx}$ being Hölder continuous with exponent $\alpha$ in $x \in [0,L]$ uniformly with respect to $t \in [0,T]$, and with $u_t$ being Hölder continuous with exponent $\alpha/2$ in $t \in$

* Corresponding author at: Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK.
*E-mail addresses:* mmmjmh@leeds.ac.uk (M.J. Huntul), amt5ld@maths.leeds.ac.uk (D. Lesnic).

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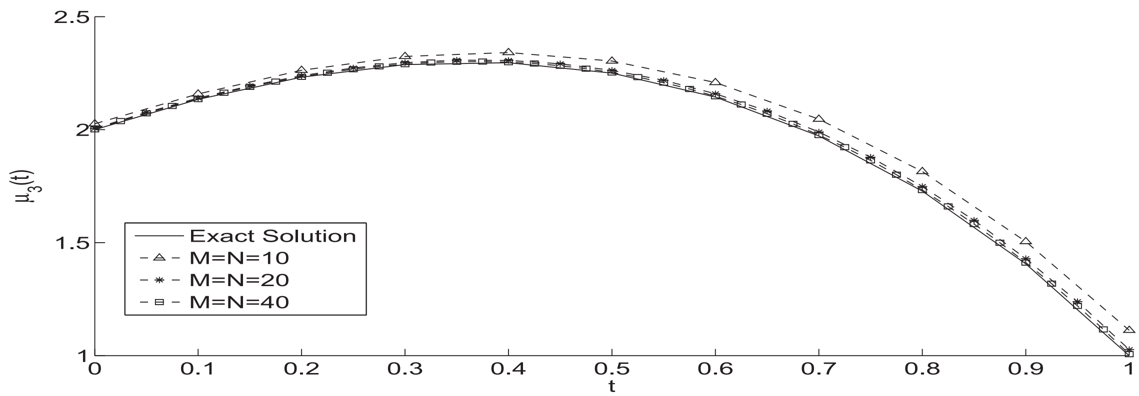


**Fig. 1.** The exact (Eq. (20)) and numerical solutions for the heat flux (Eq. (4)), for Example 1 with $M = N \in \{10,20,40\}$, for the direct problem.

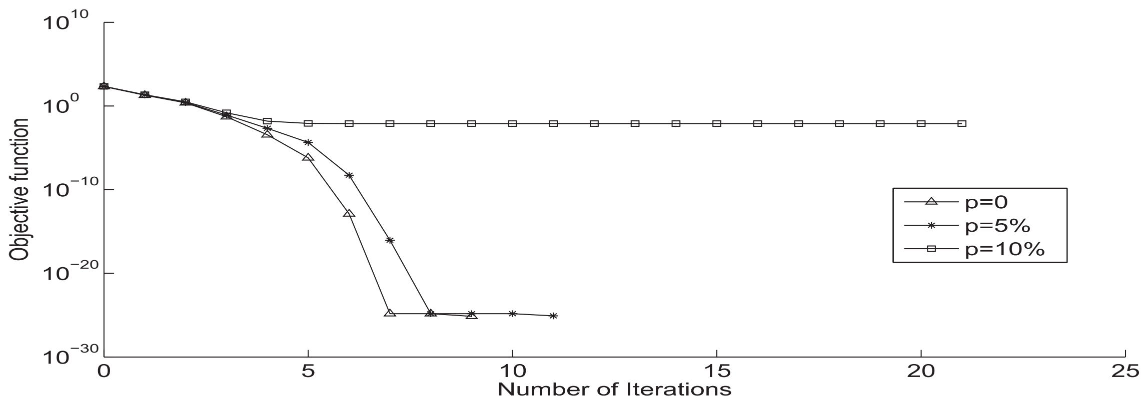


**Fig. 2.** Objective function (11), for Example 1 with $p \in \{0,5\%,10\%\}$ noise.

$[0,T]$ uniformly with respect to $x \in [0,L]$. Lower-order terms $b(x,t)\frac{\partial u}{\partial x}(x,t) + c(x,t)u(x,t)$, with $b$ and $c$ known functions, can also be added to the right-hand side of Eq. (1), with no qualitative change in both analytical and numerical analyses, [6].

## 3. Numerical solution of direct problem

In this section, we consider the direct initial boundary value problem given by Eqs. (1)–(3). We use the finite-difference method (FDM) with a Crank-Nicholson scheme, [10], which is unconditionally stable and second-order accurate in space and time. The discrete form of the direct problem is as follows. We denote $u(x_i,t_j) = u_{i,j}, a(t_j) = a_j$, and $f(x_i,t_j) = f_{i,j}$, where $x_i = i\Delta x, t_j = j\Delta t$ for $i = \overline{0, M}, j = \overline{0, N}$, and $\Delta x = \frac{L}{M}$, $\Delta t = \frac{T}{N}$. Then the problems (1)–(3) can be discretised as

$$-A_{j+1}u_{i-1,j+1} + (1 + B_{j+1})u_{i,j+1} - A_{j+1}u_{i+1,j+1}$$

$$= A_j u_{i-1,j} + (1-B_j)u_{i,j} + A_j u_{i+1,j} + \frac{\Delta t}{2}(f_{i,j} + f_{i,j+1}),$$
$$i = \overline{1, (M-1)}, \quad j = \overline{0, N}, \tag{5}$$

$$u_{i,0} = \phi(x_i), \quad i = \overline{0, M}, \tag{6}$$

$$u_{0,j} = \mu_1(t_j), \quad u_{M,j} = \mu_2(t_j), \quad j = \overline{0, N}, \tag{7}$$

where

$$A_j = \frac{(\Delta t)a_j}{2(\Delta x)^2}, \qquad B_j = \frac{(\Delta t)a_j}{(\Delta x)^2}.$$

At each time step $t_{j+1}$, for $j = \overline{0, (N-1)}$, using the Dirichlet boundary conditions Eq. (7), the above difference equation can be reformulated as a $(M-1) \times (M-1)$ system of linear equations of the form,

$$D\mathbf{u_{j+1}} = E\mathbf{u_j} + \mathbf{b^j}, \tag{8}$$

where

$$\mathbf{u_{j+1}} = (u_{1,j+1}, u_{2,j+1}, \ldots, u_{M-2,j+1}, u_{M-1,j+1})^{\mathrm{T}},$$

$$D = \begin{pmatrix} 1 + B_{j+1} & -A_{j+1} & \ldots & 0 & 0 \\ -A_{j+1} & 1 + B_{j+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 + B_{j+1} & -A_{j+1} \\ 0 & 0 & \ldots & -A_{j+1} & 1 + B_{j+1} \end{pmatrix},$$

$$E = \begin{pmatrix} 1 - B_j & A_j & \ldots & 0 & 0 \\ A_j & 1 - B_j & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 - B_j & A_j \\ 0 & 0 & \ldots & A_j & 1 - B_j \end{pmatrix},$$

and

$$\mathbf{b^j} = \begin{pmatrix} \frac{\Delta t}{2}(f_{1,j} + f_{1,j+1}) + A_j\mu_1(t_j) + A_{j+1}\mu_1(t_{j+1}) \\ \frac{\Delta t}{2}(f_{2,j} + f_{2,j+1}) \\ \vdots \\ \frac{\Delta t}{2}(f_{M-2,j} + f_{M-2,j+1}) \\ \frac{\Delta t}{2}(f_{M-1,j} + f_{M-1,j+1}) + A_j\mu_2(t_j) + A_{j+1}\mu_2(t_{j+1}) \end{pmatrix}.$$

The numerical solution for heat flux in Eq. (4) on the interval $t \in [0,T]$ is given by

$$\mu_3(t_j) = a(t_j)u_x(0, t_j) = \frac{(4u_{1,j} - u_{2,j} - 3\mu_1(t_j))a_j}{2\Delta x},$$
$$j = \overline{0, N}. \tag{9}$$

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