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Defect-correction finite element method based on Crank-Nicolson extrapolation scheme for the transient conduction-convection problem with high Reynolds number☆

Haiyan Su[a], Xinlong Feng[a,*], Yinnian He[a,b]

[a]College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, PR China
[b]Center for Computational Geosciences, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China

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ABSTRACT

A defect-correction finite element (FE) method is designed and analyzed for solving the two-dimensional (2D) transient conduction-convection problem at high Reynolds number. The method combines the merits of Crank-Nicolson (CN) extrapolation discretization and defect-correction scheme, which consists of solving a linearized problem with an added artificial viscosity term and then correcting the previous numerical solutions by a linearized defect-correction technique. The stability and optimal error estimate of the fully discrete scheme are derived. Finally, performance of the proposed method is investigated by numerical experiments.

1. Introduction

The incompressible nonstationary conduction–convection problem is one of the main system studied in fluid dynamics, which is the coupled nonlinear dynamic system of viscous incompressible flow and temperature field. For the thermodynamics view, the moving course of viscous fluid will produce quantity of heat. Thus, the fluid motion must be companied with mutual transformation of temperature, velocity and pressure for the incompressible conduction–convection problem. Therefore, research of the considered model is a subject that is full of significance. And there are numerous works devoted to the development of high efficient and promising schemes for the conduction–convection equations [1–10].

Turbulent flows are characterized by high Reynolds numbers, which occur in many processes in nature as well as in many industrial applications. But high Reynolds number may exhibit global spurious oscillations by the standard Galerkin FE method to solve the considered model for the dominance of the convection term. Then, there are many efficient numerical techniques devoted to cutting through the difficulty: the variational multiscale method in [11–14], the stabilization scheme in [15], the defect-correction method in [7,16], etc. Among them, the defect-correction is one of the popular method to deal with the problem at high Reynolds number. The main idea of the defect-correction method is to solve stabilized artificial viscosity nonlinear equations in the defect step and correct the residual by a linearized problem in the correction step for a few steps.

As a consequence, the defect-correction scheme gains extensive attentions. Layton et al. initially proposed the defect-correction method for the stationary incompressible Navier–Stokes equations with high Reynolds number in [17]. Then, Axelsson and Layton [18] applied the defect-correction method for the convection–diffusion problems. Recently, Kaya et al. [19] considered the synthesis of a subgrid stabilization method with defect-correction method for the stationary natural convection problem. Besides, a two-level defect-correction Oseen iterative stabilized FE methods for the stationary conduction–convection equations is given by Su et al. in [7].

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* Corresponding author.
E-mail addresses: shymath@163.com (H. Su), fxlmath@xju.edu.cn (X. Feng), heyn@mail.xjtu.edu.cn (Y. He).

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Furthermore, a defect-correction method for unsteady conduction–convection problems is proposed in [20] and Zhang et al. [21] investigated a defect-correction mixed FE method for steady-state natural convection problem.

In a practical application, fully implicit schemes are (almost) unconditionally stable for the nonstationary problem. However, one has to solve a system of nonlinear equations at each time step. Besides, it suffers a restriction of time step size from the stability requirement for the explicit scheme. Then, a popular approach is based on an implicit scheme for the linear term and a semi-implicit scheme or an explicit scheme for the nonlinear term. In [22], Davis et al. proposed a first-order semi-implicit scheme which using one or two steps to handle the nonlinear term in computations for the transient Navier–Stokes equations and eddy viscosity models. Obviously, high-order schemes are of more interest for the first-order schemes are not efficiently accurate for large time approximations. And CN extrapolation scheme is one of the popular stable linearized scheme with second-order accuracy. Because of its high accuracy and unconditional stability, the scheme has been widely used in many partial differential equations [23–25].

The remainder of this paper is organized as follows. In Section 2, we introduce the notations, an abstract functional setting of the transient conduction–convection problem. Mixed FE strategy and some well-known results used throughout this paper are recalled in Section 3. Defect-correction FE method based on the CN extrapolation scheme and its uniform stability, optimal error estimates are presented in Section 4. Then in Section 5, numerical experiments are shown to verify the theoretical results completely. Finally, we end with a short conclusion in Section 6.

2. Preliminaries

Let Ω be a bounded, convex and open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\Omega$. We consider the nonstationary conduction–convection equations

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \lambda jT, & \text{in} \quad \Omega\times(0,t^*], \\ \nabla\cdot u = 0, & \text{in} \quad \Omega\times(0,t^*], \\ T_t - \Delta T + \lambda u\cdot\nabla T = 0, & \text{in} \quad \Omega\times(0,t^*], \\ u(x,0) = 0, \quad T(x,0) = \xi(x), & \text{on} \quad \Omega\times\{0\}, \\ u = 0, \quad T = \gamma(x,t) & \text{in} \quad \partial\Omega\times(0,t^*], \end{cases} \tag{1}$$

where $u = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $T = T(x,t)$ the temperature, $\nu > 0$ the viscosity, which is inversely proportional to the Reynolds number, $\lambda > 0$ the Grashoff number, $\xi(x)$ and $\gamma(x,t)$ are the given function, $j = (0,1)'$ the 2D vector, t^* the given final time and $u_t = \partial u/\partial t$, $T_t = \partial T/\partial t$.

The standard weak form of the incompressible conduction–convection system Eq. (1) reads: find $(u,p,T) \in (X,M,W)$ for all $t \in (0,t^*]$ such that for all $(v,q,s) \in (X,M,W_0)$ and $T|_{\partial\Omega} = \gamma(x,t)$,

$$\begin{cases} (u_t, v) + B((u,p);(v,q)) + b(u;u,v) = \lambda(jT, v), \\ (T_t, s) + \bar{a}(T,s) + \lambda\bar{b}(u;T,s) = 0, \\ u(x,0) = 0, \quad T(x,0) = \xi(x), \end{cases} \tag{2}$$

with

$$a(u,v) = (\nabla u, \nabla v), \quad d(v,q) = (q, \operatorname{div} v), \quad \bar{a}(T,s) = (\nabla T, \nabla s),$$
$$b(u;v,w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)w, v) = \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v),$$
$$\bar{b}(u;T,s) = ((u\cdot\nabla)T, s) + \frac{1}{2}((\operatorname{div} u)T, s) = \frac{1}{2}((u\cdot\nabla)T, s) - \frac{1}{2}((u\cdot\nabla)s, T),$$
$$B((u,p);(v,q)) = \nu a(u,v) - d(v,p) + d(u,q),$$

and

$$X := \left(H_0^1(\Omega)\right)^2, \quad W := H^1(\Omega), \quad W_0 := H_0^1(\Omega),$$
$$M := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q\,dx = 0 \right\}.$$

The spaces $L^2(\Omega)^s$, $s = 1,2$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The space X is endowed with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\|\nabla u\|_0$. Standard definitions are used for the Sobolev spaces $W^{m,p}(\Omega)$, with the norm $\|\cdot\|_{m,p}$, $m,p \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$. Also, we denote by $\|\cdot\|_{L^q}$ the norm on space $L^q(\Omega)$ with $1 < q \leq \infty$.

And the trilinear forms $b(\cdot;\cdot,\cdot)$ and $\bar{b}(\cdot;\cdot,\cdot)$ satisfy

$$\begin{aligned} |b(u;v,w)| &\leq N\,\|\nabla u\|_0\,\|\nabla v\|_0\,\|\nabla w\|_0, \quad \forall u,v,w \in X, \\ |\bar{b}(u;T,s)| &\leq \bar{N}\,\|\nabla u\|_0\,\|\nabla T\|_0\,\|\nabla s\|_0, \quad \forall (u,T,s) \in (X,W,W), \end{aligned} \tag{3}$$

where

$$N = \sup_{u,v,w\in X} \frac{|b(u;v,w)|}{\|\nabla u\|_0\,\|\nabla v\|_0\,\|\nabla w\|_0}, \quad \bar{N} = \sup_{u\in X, T,s\in W} \frac{|\bar{b}(u;T,s)|}{\|\nabla u\|_0\,\|\nabla T\|_0\,\|\nabla s\|_0}.$$

3. Mixed finite element method

For $h > 0$, we consider finite-dimensional subspaces $(X_h, M_h, W_h) \subset (X,M,W)$ which are characterized by K_h, a partitioning of Ω into triangles K with the mesh size h, assumed to be uniformly regular in the usual sense. And define $W_{0h} = W_h \cap W_0$. For further details, readers can refer to Ciarlet [26]. Subsequently, c or C (with or without a subscript) will denotes a generic positive constant.

The standard FE Galerkin approximation of Eq. (2) based on (X_h, M_h, W_h) reads as follows: find $(u_h, p_h, T_h) \in (X_h, M_h, W_h)$ such that, for all $0 \leq t \leq t^*$, $T_h|_{\partial\Omega} = T^*$ (t^* is the interpolation of $\gamma(x,t)$) and $(v_h, q_h, s_h) \in (X_h, M_h, W_{0h})$,

$$\begin{cases} (u_{ht}, v_h) + B((u_h,p_h);(v_h,q_h)) + b(u_h;u_h,v_h) = \lambda(jT_h, v_h), \\ (T_{ht}, s_h) + \bar{a}(T_h,s_h) + \lambda\bar{b}(u_h;T_h,s_h) = 0, \\ u_h(x,0) = 0, \quad T_h(x,0) = T_h^0, \end{cases} \tag{4}$$

where T_h^0 is the interpolation of $\xi(x)$.

Then, we define the subspace V_h of X_h given by

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0, \forall q_h \in M_h\}.$$

According to [27], the FE space pair (X_h, M_h) and V_h satisfy the following properties:

Lemma 3.1. *Let $I_h : L^2(\Omega)^2 \to V_h$ be the standard L^2-projection. Then*

$$\|v - I_h v\|_{0,\Omega} + h\,\|\nabla(v - I_h v)\|_{0,\Omega} \leq ch^i\,\|v\|_{i,\Omega}, \quad \forall v \in H^i(\Omega) \cap V_0, \tag{5}$$

for $i = 1,2,3$, with $V_0 = \{v \in H_0^1(\Omega); \nabla\cdot v = 0\}$.

Purely for some subspace analysis, we shall often make use of the approximate divergence-free FE space V_{0h}:

$$V_{0h} = \{v_h \in V_h; (\operatorname{div} v_h, q_h) = 0, \forall q_h \in M_h\}.$$