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Mixed methods using standard conforming finite elements

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We investigate the mixed finite element method (MFEM) for solving a second order elliptic problem with a lowest order term, as might arise in the simulation of single-phase flow in porous media. We find that traditional mixed finite element spaces are not necessary when a positive lowest order (i.e., reaction) term is present. Hence, we propose to use standard conforming finite elements $Q_k \times (Q_k)^d$ on rectangles or $P_k \times (P_k)^d$ on simplices to solve for both the pressure and velocity field in d dimensions. The price we pay is that we have only sub-optimal order error estimates. With a delicate superconvergence analysis, we find some improvement for the simplest pair $Q_k \times (Q_k)^d$ with any $k \geqslant 1$, or for $P_1 \times (P_1)^d$, when the mesh is uniform and the solution has one extra order of regularity. We also prove similar results for both parabolic and second order hyperbolic problems. Numerical results using $Q_1 \times (Q_1)^2$ and $P_1 \times (P_1)^2$ are presented in support of our analysis. These observations allow us to simplify the implementation of the MFEM, especially for higher order approximations, as might arise in an *hp*-adaptive procedure.

1. Introduction

The mixed finite element method (MFEM) is often used to obtain approximate solutions to more than one unknown at the same time. For example, the MFEM is used to solve the problem of single-phase flow in porous media, described by a second order elliptic equation written as a system of two first order equations, to obtain approximations to both the pressure and Darcy velocity field simultaneously. As another example, Maxwell's equations are often solved to obtain both the magnetic and electric fields. Accordingly, we need a different finite element space for each unknown. Convergence is guaranteed if these two spaces are interrelated in that they satisfy the so-called discrete inf–sup condition, i.e., the Ladyzhenskaya–Babuška–Brezzi (LBB) condition [7,25,27].

The inf–sup condition complicates the definition of the finite element spaces. For example, in solving the single-phase flow problem, many complicated mixed finite element spaces have been proposed such as those of Raviart–Thomas–Nedelec [26,24], Brezzi–Douglas–Marini [5], Brezzi–Douglas–Duràn–Fortin [6], Chen–Douglas [12], Brezzi–Fortin–Marini [8], and Arbogast–Wheeler [3]. More details can be consulted the books by Brezzi–Fortin [7] and Roberts–Thomas [27] and references therein. Due to the complicated nature of these mixed spaces, usually only the lowest order spaces are used in practical computations. Furthermore, some postprocessing is needed to visualize the numerical solution, since the degrees of freedom for the MFEM are not nodal-based. Such postprocessing further complicates the implementation of the MFEM. Hence, it would be desirable in some cases to be able to use standard nodal basis finite element spaces in the MFEM. Such efforts have been carried out in the engineering community (e.g., [17,29]).

In this paper, we approximate a second order elliptic equation, written in mixed form, by applying the standard conforming finite element spaces, $Q_k \times (Q_k)^d$ on rectangles or $P_k \times (P_k)^d$ on simplices in dimension $d = 2, 3$, where Q_k and P_k are continuous piecewise polynomials of degree k in each Cartesian variable separately for Q_k, and of total degree k for P_k. A careful investigation shows that these spaces can be successfully used to solve for both the pressure and velocity field when a reaction term is present, i.e., when a uniformly positive zeroth order term appears in the equation. Because the inf–sup condition is violated, however, we have sub-optimal convergence properties, losing a single power of the mesh spacing h. On simplicial meshes, these spaces are smaller in their number of degrees of freedom than the Raviart–Thomas spaces for similar accuracy. In the rectangular mesh case, again for similar accuracy, these spaces are slightly larger than the Raviart–Thomas spaces.

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However our proposed spaces are much simpler to implement, especially when higher order spaces are desired, or when an *hp*-adaptive refinement procedure is implemented.

A locally conservative variant is easy to define, in which the scalar variable is approximated by a discontinuous space of Q_k or P_k piecewise polynomials. In fact, we can use P_k on both rectangular and simplicial meshes.

The convergence result is sub-optimal, however, we can recover one-half power of h on uniform grids for $Q_k \times (Q_k)^d$ with any $k \geqslant 1$, or for $P_1 \times (P_1)^d$. Moreover, we recover a full power of h, and thereby obtain optimal accuracy, when the problem has periodic boundary conditions, as might arise, e.g., from a cell problem in homogenization. This analysis is based on Lin's integral identity technique developed in the early 1990s (see, e.g., [20,23,21,33]) for proving general finite element method superconvergence. More details and applications of this technique can be found in the superconvergence books [9,22].

Finally, we extend our results to parabolic and second order hyperbolic problems. To the best of our knowledge, no previous reference has pointed out the interesting properties we note in this paper for the standard spaces in a mixed context when a time derivative or uniformly positive zeroth order term appears. Because of their simplicity and convergence properties, they seem to be competitive with, and perhaps the better choice than, corresponding Raviart–Thomas spaces when (1) simplicial meshes are used, (2) problems with periodic boundary conditions are approximated with uniform rectangular grids, (3) higher order approximations are desired, and (4) when an *hp*-adaptive refinement procedure is implemented.

The rest of this paper is organized as follows. In Section 2, we formulate the MFEM for the elliptic single-phase flow problem using $Q_k \times (Q_k)^d$ and $P_k \times (P_k)^d$ spaces. Existence and uniqueness of the system is proved, and error estimates are obtained. We pay special attention to the size of our mixed spaces, and compare to some which satisfy the inf–sup condition. Our improved error analysis is also presented here. In Sections 3 and 4, we generalize the results to parabolic and hyperbolic problems, respectively. In Section 5, we provide some numerical examples to illustrate our method and confirm our theoretical analysis. Section 6 concludes the paper.

2. The elliptic problem

Let $\Omega \subset \boldsymbol{R}^d$, d = 2, 3, be an open Lipschitz polygon or polyhedron. For simplicity, we consider the single-phase flow problem

$$-\nabla \cdot [D(\boldsymbol{x})(\nabla p - \boldsymbol{g}(\boldsymbol{x}))] + \alpha(\boldsymbol{x})p(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } \Omega, \tag{1}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where the reaction coefficient $\alpha(\boldsymbol{x}) \geqslant \alpha_{\min} > 0$ and $D(x) \geqslant D_{\min} > 0$. (Other boundary conditions, including non-homogeneous Dirichlet and Neumann ones, could be treated by the techniques used in this paper, but we have restricted ourselves to the homogeneous Dirichlet condition for expository purposes.) Introduce $\boldsymbol{u} = -D(\nabla p - \boldsymbol{g})$ and transform (1) and (2) to the standard mixed form: Find $p \in L^2(\Omega)$ and $\boldsymbol{u} \in H(\mathrm{div};\Omega)$ such that

$$(\alpha p, w) + (\nabla \cdot \boldsymbol{u}, w) = (f, w) \quad \forall w \in L^2(\Omega), \tag{3}$$

$$-(p, \nabla \cdot \boldsymbol{v}) + (\beta \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in H(\mathrm{div};\Omega), \tag{4}$$

where $\beta = D^{-1}$ and $(\cdot,\cdot)_\omega$ denotes the $L^2(\Omega)$ inner-product (we omit ω if $\omega = \Omega$).

2.1. Discretization

Let T_h be a conforming quasi-uniform finite element partition of Ω by either rectangular or simplicial elements of maximal diameter h. We propose the mixed finite element approximation: Find $p_h \in W^k_{h,0}$ and $\boldsymbol{u}_h \in \boldsymbol{V}^k_h$ such that

$$(\alpha p_h, w_h) + (\nabla \cdot \boldsymbol{u}_h, w_h) = (f, w_h) \quad \forall w_h \in W^k_{h,0}, \tag{5}$$

$$-(p_h, \nabla \cdot \boldsymbol{v}_h) + (\beta \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{g}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}^k_h, \tag{6}$$

where, for $k \geqslant 1$, the mixed finite element spaces are $W^k_{h,0} = W^k_h \cap H^1_0(\Omega)$, and

$$W^k_h = \{w \in C^0(\Omega) : w|_R \in W^k_h(E), \forall E \in T_h\}, \tag{7}$$

$$\boldsymbol{V}^k_h = \{\boldsymbol{v} \in (C^0(\Omega))^d : \boldsymbol{v}|_E = (W^k_h(E))^d, \forall E \in T_h\}, \tag{8}$$

and where $W^k_h(E) = Q_k(E)$ or $P_k(E)$ for rectangular or simplicial elements, respectively. Note that our mixed spaces $\boldsymbol{V}^k_h$ are the simplest $(H^1(\Omega))^d$ elements, and not any of the standard mixed finite element spaces [7,27].

If the standard nodal basis is used, we can interleave the pressure and velocity unknowns to obtain a linear system with a block-structured matrix. The matrix has the standard stencil in terms of its blocks, and each block is $(d+1) \times (d+1)$. If we separate the pressure and velocity unknowns, we obtain a more standard saddle point linear system. In any case, as with all mixed methods, we do not have a simple positive definite system, and some care must be exercised in solving the linear system. (In our numerical results below, we used a direct solver.)

Remark 2.1. We note that we could have taken

$$W^k_{h,0} = W^{D,k}_{h,0} = \{w : w|_R \in W^k_h(E), \forall E \in T_h, w = 0 \text{ on } \partial\Omega\};$$

that is, we could relax the continuity of the pressure approximating space. Moreover, in this case, we could replace $Q_k(E)$ by $P_k(E)$ on rectangles. Our sub-optimal order error estimates below would continue to hold, but not our improved error estimates. However, this form of the method would satisfy the local mass conservation principle. That is, in this case, (5) implies that on each element $E \in T_h$,

$$(\alpha p_h, 1)_E + (\nabla \cdot \boldsymbol{u}_h, 1)_E = (\alpha p_h, 1)_E + (\boldsymbol{u}_h \cdot \boldsymbol{n}, 1)_{\partial E} = (f, 1)_E,$$

so that the net flux through ∂E, $(\boldsymbol{u}_h \cdot \boldsymbol{n}, 1)_{\partial E}$, is exactly related to the net external sources $(f,1)_E$ and the reaction or accumulation $(\alpha p_h, 1)_E$ acting over E. This is an important property in certain applications (see, e.g., [14,28,1]).

Remark 2.2. For Galerkin formulations, the Dirichlet boundary condition (BC) is essential and the Neumann BC is natural, whereas mixed methods have the opposite behavior. The Neumann boundary condition, being essential, is easily incorporated by fixing the normal components of u and v to the data and zero, respectively, and taking p in W^k_h. The Dirichlet BC is natural. For the non-homogeneous case $p = p_D$ on $\partial\Omega$, we would modify (4) to read

$$-(p, \nabla \cdot \boldsymbol{v}) + (\beta \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v}) - (p_D, \boldsymbol{v} \cdot \boldsymbol{n}) \quad \forall \boldsymbol{v} \in H(\mathrm{div};\Omega),$$

and the method similarly. This BC is imposed naturally with both p_h and w_h in W^k_h. However, we have a larger finite element space (W^k_h versus $W^k_{h,0}$), and the BC is not set exactly. Therefore, we chose above to impose the Dirichlet BC as an essential BC. That is, we take p in W^k_h such that p agrees with p_D on $\partial\Omega$, and we restrict w_h to $W^k_{h,0}$.

2.2. Sub-optimal order error estimates

Let $\|\cdot\|_{k,\omega}$ denote the $H^k(\omega)$-norm, and let $|\cdot|_{k,\omega}$ be the $H^k(\omega)$-semi-norm of highest derivatives only, wherein we omit ω if it is Ω. For function w, let w_I denote the standard Q_k or P_k interpolant. Then we have the well-known interpolation estimate