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Thermo-mechanical contact problems on non-matching meshes

S. Hüeber, B.I. Wohlmuth *

Institut für Angewandte Analysis und Numerische Simulation (IANS), Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany

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ABSTRACT

Non-matching meshes and domain decomposition techniques based on Lagrange multipliers provide a flexible and efficient discretization technique for variational inequalities with interface constraints. Although mortar methods are well analyzed for variational inequalities, its application to dynamic thermo-mechanical contact problems with friction is still a field of active research. In this work, we extend the mortar approach for dynamic contact problems with Coulomb friction to the thermo-mechanical case. We focus on the discretization and on algorithmic aspects of dynamic effects such as frictional heating and thermal softening at the contact interface. More precisely, we generalize the mortar concept of dual Lagrange multipliers to non-linear Robin-type interface conditions and apply local static condensation to eliminate the heat flux. Numerical examples in the two-dimensional and the three-dimensional setting illustrate the flexibility of the discretization on non-matching meshes.

1. Introduction

The numerical simulation of frictional contact problems is still a challenging task and plays an important role in many industrial applications. Mortar techniques became a promising discretization method for such type of problems involving non-matching meshes, see, e.g. [9,21,3,5]. Recently, a lot of work has been done to generalize these concepts to dynamic contact problems including friction and large deformations. Due to the sliding of the different bodies, most of the frictional work results in the generation of heat. This observation motivates the extension of the mortar method to thermo-mechanical dynamic contact problems including frictional heating and thermal softening effects at the contact interface.

The mortar method is a hybrid formulation in space. The displacement and the temperature field enter as primal variable, whereas the contact stress and the thermal flux at the contact interface are the dual variables. Mathematically speaking, the dual variables, also denoted as Lagrange multipliers, enforce the interface conditions. Here, we have two types of constraints: the non-penetration condition and the friction law for the mechanical unknowns and the generation of heat as well as the flow condition for the thermal variables. The main focus of this paper is on the treatment of the Robin-type thermal interface condition within the mortar framework. Due to the use of biorthogonal basis functions for the mechanical dual variable, the mechanical interface constraints at the contact zone decouple for each node. Although the mortar approach can be seen as a segment-to-segment formulation, the possible decoupling results in a node-to-segment approach which can be handled more easily from the numerical point of view. In contrast, a straightforward application of the concept of biorthogonality to Robin-type constraints does not decouple the nodes. To benefit from static condensation, we introduce a stable operator which can be interpreted as mass lumping at the interface.

For the formulation of the linear thermo-elastic constitutive equations, we follow [2,17]. A more general thermo-plasticity formulation is given in [15]. The extension to dynamic thermo-mechanical contact mechanics can be found, e.g. in [10,13,12] and is also considered in the textbooks [20,17]. For the modeling of the contact heat flux, we use the linear model proposed in [13]. More general models can be found in [18]. For the considered formulation with a coefficient of friction depending on the temperature we refer to [13,12].

We start with the introduction of the constitutive equations and the interface conditions in Section 2. Section 3 presents the time discretization based on the midpoint scheme. The mortar formulation and the space discretization can be found in Section 4, followed by the resulting algebraic formulation in Section 5. Some comments on the applied numerical algorithm to solve the arising non-linear equations are given in Section 6. The last section shows numerical examples both in the two- and three-dimensional setting and illustrates the flexibility of the considered discretization.

* Corresponding author.
E-mail addresses: hueeber@ians.uni-stuffgart.de (S. Hüeber), wohlmuth@ians.uni-stuffgart.de, seka7@ians.uni-stuttgart.de (B.I. Wohlmuth).

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2. Problem formulation for linear thermo-elasticity

We consider two bodies in their reference configuration $\Omega^i \subset \mathbb{R}^d, i \in \{m,s\}$, with the dimension $d = 2,3$. The superscript s stands for the slave body and m for the master body, as it is common in the framework of mortar techniques. We are interested in the displacement field $\boldsymbol{u}^i(\boldsymbol{x},t)$ and the temperature $\theta^i(\boldsymbol{x},t)$ for $(\boldsymbol{x},t) \in \Omega^i \times (0,T)$, where $(0,T)$ is the given time interval. The local balance of momentum is given by

$$\varrho^i \ddot{\boldsymbol{u}}^i - \mathrm{Div}(\boldsymbol{P}^i) = \boldsymbol{f}^i \quad \text{in } \Omega^i \times (0,T). \tag{2.1}$$

Here we denote by ϱ^i the mass density of the body Ω^i and by $\boldsymbol{P}^i$ the first Piola–Kirchhoff stress tensor. The vector $\boldsymbol{f}^i$ describes the given body forces acting on Ω^i. In the case of linear thermo-elasticity, the stress tensor is given by

$$\boldsymbol{P}^i := \lambda^i \mathrm{tr}(\boldsymbol{\varepsilon}^i)\mathrm{Id} + 2\mu^i \boldsymbol{\varepsilon}^i - dK^i \alpha^i (\theta^i - \theta_0)\mathrm{Id}, \tag{2.2}$$

see [2,17], where the linearized strain tensor satisfies $\boldsymbol{\varepsilon}^i := \frac{1}{2}(\nabla \boldsymbol{u}^i + (\nabla \boldsymbol{u}^i)^\top)$. The Lamé parameters are obtained by $\lambda^i := (E^i \nu^i)/((1+\nu^i)(1-2\nu^i))$ and $\mu^i := E^i/(2(1+\nu^i))$ with Young's modulus $E^i > 0$ and the Poisson ratio $\nu^i \in (0, 0.5)$. The bulk modulus is given by $K^i := \lambda^i + \frac{2}{d}\mu^i$. Moreover, $\mathrm{tr}(\cdot)$ denotes the trace operator and Id the identity tensor in $\mathbb{R}^d$. θ_0 is a reference temperature at which the bodies are stress free, and the factor α^i denotes the thermal expansion coefficient of the material of body Ω^i. We remark that the relation between the relative temperature $\theta^i - \theta_0$ and the stress is given by the stress–temperature tensor, which reduces to $dK^i\alpha^i\mathrm{Id}$, due to the assumed isotropy.

In addition, we have to consider the heat conduction equation which results from the first law of thermodynamics:

$$c^i_\theta \dot{\theta}^i = -\mathscr{H}^i - \mathrm{div}(\boldsymbol{q}^i) + r^i \quad \text{in } \Omega^i \times (0,T) \tag{2.3}$$

with the specific heat capacity c^i_θ of the body Ω^i. The term $\mathscr{H}^i$ denotes the heating from the Joule effect, and the prescribed heat source term is given by r^i. Following [17], we get for linearized thermo-elasticity:

$$\mathscr{H}^i := d\alpha^i K^i \theta_0 \mathrm{div}(\dot{\boldsymbol{u}}^i). \tag{2.4}$$

Due to the classical Fourier's law, the heat flux is given by

$$\boldsymbol{q}^i = -\kappa^i \nabla \theta^i \tag{2.5}$$

with the thermal conductivity $\kappa^i > 0$. We remark that in a more general setting, κ^i is a tensor which is positive semi-definite as a consequence of the second law of thermodynamics.

To formulate the initial boundary value problem, we divide the boundary in the reference configuration $\Gamma^i := \partial\Omega^i$ of the domain Ω^i into three nonoverlapping open subsets Γ^i_D, Γ^i_N and Γ^i_c with $\overline{\Gamma}^i_D \cup \overline{\Gamma}^i_N \cup \overline{\Gamma}^i_c = \Gamma^i$. In addition, we have two subsets Γ^i_θ and Γ^i_q such that $\overline{\Gamma}^i_\theta \cup \overline{\Gamma}^i_q = \overline{\Gamma}^i_D \cup \overline{\Gamma}^i_N$ and $\Gamma^i_\theta \cap \Gamma^i_q = \emptyset$. Defining the linear stress tensor $\boldsymbol{\sigma}^i := \lambda^i \mathrm{tr}(\boldsymbol{\varepsilon}^i)\mathrm{Id} + 2\mu^i\boldsymbol{\varepsilon}^i$, we summarize (2.1)–(2.5) and formulate the initial boundary value problem

$$\varrho^i \ddot{\boldsymbol{u}}^i - \mathrm{Div}(\boldsymbol{\sigma}^i) + dK^i\alpha^i\nabla\theta^i = \boldsymbol{f}^{\,i} \quad \text{in } \Omega^i \times (0,T), \tag{2.6a}$$
$$c^i_\theta \dot{\theta}^i + d\alpha^i K^i \theta_0 \mathrm{div}(\dot{\boldsymbol{u}}^i) - \mathrm{div}(\kappa^i \nabla\theta^i) = r^i \quad \text{in } \Omega^i \times (0,T), \tag{2.6b}$$

where we assume θ_0 to be constant on $\Omega^s \cup \Omega^m$. As initial conditions, we set

$$\begin{aligned} &\boldsymbol{u}^i(\boldsymbol{x},0) = \boldsymbol{0} \quad \text{in } \Omega^i,\\ &\dot{\boldsymbol{u}}^i(\boldsymbol{x},0) = \boldsymbol{v}^i_0(\boldsymbol{x}) \quad \text{in } \Omega^i,\\ &\theta^i(\boldsymbol{x},0) = \theta^i_0(\boldsymbol{x}) \quad \text{in } \Omega^i, \end{aligned} \tag{2.7}$$

where $\boldsymbol{v}^i_0(\boldsymbol{x})$ denotes the initial velocity of the body Ω^i and $\theta^i_0(\boldsymbol{x})$ its initial temperature. To obtain a well defined system, we have to specify suitable boundary conditions for the displacement and the temperature

$$\begin{aligned} &\boldsymbol{u}^i = \boldsymbol{u}^i_D \quad \text{on } \Gamma^i_D \times (0,T),\\ &\boldsymbol{P}^i\boldsymbol{n}^i_0 = \boldsymbol{t}^i_N \quad \text{on } \Gamma^i_N \times (0,T),\\ &\theta^i = \theta^i_D \quad \text{on } \Gamma^i_\theta \times (0,T),\\ &\boldsymbol{q}^i\boldsymbol{n}^i_0 = q^i_N \quad \text{on } \Gamma^i_q \times (0,T), \end{aligned} \tag{2.8}$$

where $\boldsymbol{n}^i_0$ denotes the outward unit normal vector in the reference configuration on Γ^i. To define the interface conditions on the possible contact boundary Γ^i_c, we follow [13,12] and start with introducing for each point $\boldsymbol{x} \in \Gamma^s_c$ the normal vector to the current configuration $\boldsymbol{n} := \boldsymbol{n}^s$. A predefined relation between the points of the possible contact zones Γ^i_c is constructed by a smooth mapping $R_t(\boldsymbol{x}) : \Gamma^s_c \to \Gamma^m_c$ satisfying $R_t(\Gamma^s_c) \subset \Gamma^m_c$ for all $t \in (0,T)$. The current position $\boldsymbol{\varphi}^s(\boldsymbol{x},t) := \boldsymbol{x} + \boldsymbol{u}^s(\boldsymbol{x},t) \in \Gamma^s_{c,t}$ of the reference point $\boldsymbol{x} \in \Gamma^s_c$ is projected onto the current master boundary $\Gamma^m_{c,t}$ in the direction of the current normal $\boldsymbol{n}(\boldsymbol{x},t)$; the reference point corresponding to the projection $\boldsymbol{\varphi}^m(R_t(\boldsymbol{x}),t) \in \Gamma^m_{c,t}$ defines $R_t(\boldsymbol{x})$. We assume that this mapping is well defined. Denoting the current gap at a point $\boldsymbol{x} \in \Gamma^s_c$ by

$$g(\boldsymbol{x},t) := (\boldsymbol{\varphi}^s(\boldsymbol{x},t) - \boldsymbol{\varphi}^m(R_t(\boldsymbol{x}),t))\boldsymbol{n}(\boldsymbol{x},t), \tag{2.9}$$

the contact conditions on Γ^s_c reads

$$g(\boldsymbol{x},t) \leqslant 0, \quad p_n(\boldsymbol{x},t) \geqslant 0, \quad g(\boldsymbol{x},t)p_n(\boldsymbol{x},t) = 0 \tag{2.10}$$

for all $t \in (0,T)$, where the normal part of the contact stress is given by $p_n := \boldsymbol{p}_c\boldsymbol{n}$ for the total contact stress $\boldsymbol{p}_c := -\boldsymbol{P}^s\boldsymbol{n}^s_0$. For the action–reaction principle, we have to satisfy for all $t \in (0,T)$ the condition

$$\boldsymbol{p}_c = -\boldsymbol{P}^s(\boldsymbol{x},t)\boldsymbol{n}^s_0 = \boldsymbol{P}^m(R_t(\boldsymbol{x}),t)\boldsymbol{n}^m_0 \quad \text{on } \Gamma^s_c. \tag{2.11}$$

We note that this equality is restricted to the linear theory, otherwise the Jacobian of $R_t(\cdot)$ has to be taken into account. Using the normal component p_n of the contact stress $\boldsymbol{p}_c$, we define its tangential part by $\boldsymbol{p}_\tau := \boldsymbol{p}_c - p_n\boldsymbol{n}$. Furthermore, we introduce the relative displacement along the possible contact boundary

$$[\boldsymbol{u}(\boldsymbol{x},t)] := \boldsymbol{u}^s(\boldsymbol{x},t) - \boldsymbol{u}^m(R_t(\boldsymbol{x}),t) \quad \text{for } (\boldsymbol{x},t) \in \Gamma^s_c \times (0,T) \tag{2.12}$$

and split it into normal part $[\boldsymbol{u}]_n := [\boldsymbol{u}]\boldsymbol{n}$ and tangential part $[\boldsymbol{u}]_\tau := [\boldsymbol{u}] - [\boldsymbol{u}]_n\boldsymbol{n}$. Defining the maximal temperature of the two contact interfaces

$$\bar{\theta}_c(\boldsymbol{x},t) := \max\{\theta^s(\boldsymbol{x},t), \theta^m(R_t(\boldsymbol{x}),t)\} \quad \text{for } (\boldsymbol{x},t) \in \Gamma^s_c \times (0,T) \tag{2.13}$$

and using the Euclidean norm $\|\cdot\|$, we can formulate Coulomb's friction law as

$$\begin{cases} \|\boldsymbol{p}_\tau\| \leqslant \mathfrak{F}(\bar{\theta}_c)|p_n|, \\ \|\boldsymbol{p}_\tau\| < \mathfrak{F}(\bar{\theta}_c)|p_n| \Rightarrow [\dot{\boldsymbol{u}}]_\tau = 0, \\ \|\boldsymbol{p}_\tau\| = \mathfrak{F}(\bar{\theta}_c)|p_n| \Rightarrow \exists \beta : \boldsymbol{p}_\tau = \beta^2[\dot{\boldsymbol{u}}]_\tau, \end{cases} \quad \text{on } \Gamma^s_c \times (0,T), \tag{2.14}$$

where the following temperature dependent coefficient of friction $\mathfrak{F}(\bar{\theta}_c) \geqslant 0$ is used:

$$\mathfrak{F}(\bar{\theta}_c) := \mathfrak{F}_0 \frac{(\bar{\theta}_c - \theta_{\mathrm{dam}})^2}{(\theta_{\mathrm{dam}} - \theta_{\mathrm{ref}})^2}. \tag{2.15}$$

In this definition, $\mathfrak{F}_0$ is the static coefficient of friction at the given reference temperature θ_{ref}, and θ_{dam} is a damage temperature on the interface. Typically, θ_{dam} is related to the temperature at which frictional stress is no longer due to solid shearing effects but is generated by viscous shear of a molten film on the contact interface. It can be taken as the lowest melting temperature of the two bodies in contact, see [12]. Since $\bar{\theta}_c < \theta_{\mathrm{dam}}$, we have $\mathfrak{F}'(\bar{\theta}_c) \leqslant 0$ and $\lim_{\bar{\theta}_c \to \theta_{\mathrm{dam}}} \mathfrak{F}(\bar{\theta}_c) = 0$. Therefore, (2.15) shows a thermal softening effect.

The conditions for the heat flux $q^i_c := \boldsymbol{q}^i\boldsymbol{n}^i$ across the possible contact interface Γ^i_c can be written as