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A simple strategy to assess the error in the numerical wave number of the finite element solution of the Helmholtz equation ☆

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ABSTRACT

The standard approach for goal oriented error estimation and adaptivity uses an error representation via an adjoint problem, based on the linear functional output representing the quantity of interest. For the assessment of the error in the approximation of the wave number for the Helmholtz problem (also referred to as dispersion or pollution error), this strategy cannot be applied. This is because there is no linear extractor producing the wave number from the solution of the acoustic problem. Moreover, in this context, the error assessment paradigm is reverted in the sense that the exact value of the wave number, κ, is known (it is part of the problem data) and the effort produced in the error assessment technique aims at obtaining the numerical wave number, κ_H, as a postprocess of the numerical solution, u_H. The strategy introduced in this paper is based on the ideas used in the a priori analysis. A modified equation corresponding to a modified wave number κ_m is introduced. Then, the value of κ_m such that the modified problem better accommodates the numerical solution u_H is taken as the estimate of the numerical wave number κ_H. Thus, both global and local versions of the error estimator are proposed. The obtained estimates of the dispersion error match the a priori predicted dispersion error and, in academical examples, the actual values of the error in the wave number.

1. Introduction

The numerical simulation of acoustic problems requires an accurate answer to properly predict their performance. In the low frequency range domain the finite element method (FEM) is a standard tool for solving the acoustic equations. In the medium and high frequency ranges the end-user should be concerned by the errors associated with the numerical discretization. In practice, two components of the error are clearly identified in this framework: interpolation error and pollution error. The classical interpolation error decays with the mesh size as predicted by standard a priori error estimates. The behavior of the pollution error is more complex: the convergence rate predicted by the a priori estimates depends on the range where the mesh size lies (relative to the wavelength) [1].

In practice, the end-user of a finite element acoustic computation is concerned with the accuracy of the solution in terms of the dispersion, the error committed in the evaluation of the wave number, κ. Paradoxically, this is not because the value of κ is a quantity of interest that has to be evaluated accurately. In fact, the exact value of κ is known a priori as part of the problem data. The overall quality of the numerical solution is however associated with the error in the approximation of κ.

The standard approach for goal oriented error estimation and adaptivity is based on the representation of the error in a quantity of interest obtained using an adjoint problem [14,17]. The solution of the adjoint problem is also denoted extractor and the corresponding error representation combines the extractor and the original solution. Thus, the error assessment for the quantity of interest is reduced to assess the error in energy norm of this auxiliary problem. This strategy cannot be used when the quantity to be assessed is the wave number. This is because there is not a proper extractor associated with this quantity, κ. Moreover, as already noted, the exact value of κ is a priori known. This reverts the final goal of the error assessment technique. The target of the error estimation strategies is in standard cases to find a better approximation than the one provided by the numerical solution, u_H, and then compare them. In the present situation, this is somehow reverted to find the actual approximation of the quantity of interest provided by u_H, say κ_H, and to compare it with the exact value κ. Summarizing, assessing the error in κ requires a complete different paradigm. The quality of the solution is assessed via the approximation of a

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quantity which is exactly known. The numerical wave number κ_H is unknown and has to be evaluated.

The first problem to face is to find a proper definition for κ_H. Heuristically, the *wavelength* of the approximate solution is the distance of two consecutive local maxima (or minima). Although this represents a precise definition for 1D waves, it cannot be easily generalized to higher dimensions. Moreover, it cannot be converted into an explicit functional output of the numerical solution. One definition for κ_H is implicitly used in a priori analysis, based on the idea of fitting the numerical solution into a modified equation. Here, this concept is extended such that it can be exploited in a posteriori error assessment setting.

Namely, this paper introduces a technique to assess the value of κ_H based on finding the wave number of a modified problem which better accommodates the numerical solution u_H. This approach is inspired by the a priori estimates developed in [12].

The idea is also extended to find a local indicator of the error in the wavelength. This local quantity is assumed to measure the ability of the local discretization (in a given portion of the domain) to properly capture the wavelength. The possible use of this information to adapt the mesh and reduce the overall error is beyond the scope of this paper but is part of the work in progress.

The remainder of the paper is structured as follows. Section 2 introduces the notation presenting the problem to be solved, the finite element formulation and the concepts of dispersion and pollution effect in this type of problem. The basic lines of the a priori analysis performed in [12] are briefly sketched in Section 3. Then, Section 4 is devoted to introduce the a posteriori technique proposed to assess the error in the wave number. A local version of the estimate providing a spatial error distribution for adaptive purposes is introduced in Section 5. Finally, Section 6 contains numerical examples showing the good behavior of the proposed technique both in academic and practical examples.

2. Problem statement

2.1. Acoustic modeling: the Helmholtz equation

The presentation and notation introduced by Ihlenburg [11] is followed in the remainder of this section.

The transient acoustic problem consists in obtaining the unknown pressure field $P(\mathbf{x},t)$, taking values for $\mathbf{x}\in\Omega\subset\mathbb{R}^d$ (d being the dimension in space, d = 1, 2 or 3). The field $P(\mathbf{x},t)$ is the solution of the following partial differential equation:

$$\Delta P = \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{1}$$

where c is the speed of sound in the medium.

The pressure time dependency is eliminated assuming a harmonic behavior and selecting an angular frequency ω, namely

$$P(\mathbf{x},t) = u(\mathbf{x})\exp(i\omega t), \tag{2}$$

where $u(\mathbf{x})$ is the complex spatial distribution of the acoustic pressure and i the imaginary unit. Substituting (2) into (1), the wave equation reduces to the Helmholtz equation:

$$\Delta u + \kappa^2 u = 0, \tag{3}$$

where $\kappa := \omega/c$ stands for the wave number.

The physical pressure is the real part of the complex unknown u. The velocity $\mathbf{v}$ is proportional to the gradient of pressure:

$$\nabla u = -i\rho c\kappa\mathbf{v}, \tag{4}$$

where ρ is the density of the fluid.

A complete definition of the Boundary Value Problem to be solved requires adding to Eq. (3) a proper set of boundary conditions. For interior acoustic problems, three types of boundary conditions are considered: Dirichlet, Neumann and Robin (or mixed).

The Dirichlet boundary conditions prescribe values of the pressure on part of the boundary, say $\Gamma_D\subset\partial\Omega$, where u is prescribed to be equal to a given value $\bar{u}$, that is

$$u = \bar{u} \quad \text{on } \Gamma_D. \tag{5}$$

On the Neumann part of the boundary $\Gamma_N\subset\partial\Omega$ the normal component of the velocity $\mathbf{v}$ is prescribed to be equal to $\bar{v}_n$, namely

$$\frac{\partial u}{\partial \mathbf{n}} = -i\rho c\kappa\bar{v}_n \quad \text{on } \Gamma_N. \tag{6}$$

The prescribed value $\bar{v}_n$ corresponds to the normal velocity of a vibrating wall producing the sound that propagates within the medium.

Finally, on the Robin part of the boundary $\Gamma_R\subset\partial\Omega$ the velocity is imposed to be proportional to the pressure, that is

$$\frac{\partial u}{\partial \mathbf{n}} = -i\rho c\kappa A_n u \quad \text{on } \Gamma_R, \tag{7}$$

where the coefficient A_n is the admittance and represents the structural damping. This type of boundary conditions is associated with absorbing walls. For $A_n = 0$ it coincides with the homogeneous Neumann boundary condition, standing for a perfectly reflecting panel. For particular case of plane waves, the value $A_n = 1/\rho c$ describes a fully absorbent panel.

In order to get a well posed Boundary Value Problem, the three parts of the boundary must cover the whole boundary, that is $\partial\Omega = \overline{\Gamma_D\bigcup\Gamma_N\bigcup\Gamma_R}$.

The weak form of the Boundary Value Problem defined by Eqs. (3), (5)–(7) is readily expressed in its weak form using the corresponding natural functional spaces. The space for the trial functions is $U = \{u\in H^1(\Omega), u|_{\Gamma_D} = \bar{u}\}$ while the space for the test functions is $V = \{v\in H^1(\Omega), v|_{\Gamma_D} = 0\}$, $H^1(\Omega)$ being the standard Hilbert space of square integrable functions with square integrable first derivatives.

Thus, the weak form of the problem reads: find $u\in U$ such that

$$a(u,v) = l(v) \quad \forall v\in V, \tag{8}$$

where the bilinear and linear forms are defined as follows:

$$a(u,v) := \int_\Omega \nabla u\cdot\nabla\tilde{v}\,d\Omega - \int_\Omega \kappa^2 u\tilde{v}\,d\Omega + \int_{\Gamma_R} i\rho c\kappa A_n u\tilde{v}\,d\Gamma \quad \text{and}$$

$$l(v) := -\int_{\Gamma_N} i\rho c\kappa\bar{v}_n\tilde{v}\,d\Gamma$$

and the symbol $\tilde{\cdot}$ denotes the complex conjugate.

2.2. Finite element approximation

The discrete counterparts of U and V are the finite element spaces $U_H\subset U$ and $V_H\subset V$ associated with a mesh of characteristic element size H. Thus, the discrete finite element solution is the function $u_H\in U_H$ such that

$$a(u_H, v_H) = l(v_H) \quad \forall v_H\in V_H. \tag{9}$$

The finite element solution u_H is expressed in terms of the basis-functions N_j spanning U_H:

$$u_H = \sum_{j=1}^{n} N_j u_j = \mathbf{N}\mathbf{u}_H, \tag{10}$$

where u_j, for $j = 1,2,\dots,n$, are the complex nodal values, $\mathbf{N} = [N_1, N_2, \dots, N_n]$ and $\mathbf{u}_H^{\mathrm{T}} = [u_1, u_2, \dots, u_n]$.

The matrix form of (9) reads

$$(\mathbf{K}_H + i\rho c\kappa A_n\mathbf{C}_H - k^2\mathbf{M}_H)\mathbf{u}_H = -i\rho c\kappa\mathbf{f}_H, \tag{11}$$