Comput. Methods Appl. Mech. Engrg. 198 (2008) 459–474

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Stochastic fracture mechanics by fractal finite element method

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ARTICLE INFO

Article history:
Received 18 February 2008
Received in revised form 25 August 2008
Accepted 28 August 2008
Available online 9 September 2008

Keywords:
Probabilistic fracture mechanics
Stochastic fractal finite element method
Shape sensitivity analysis
Stress intensity factor
J-integral
Probability of failure

ABSTRACT

This paper presents stochastic fracture mechanics analysis of linear-elastic cracked structures subjected to mixed-mode (modes I and II) loading conditions using fractal finite element method (FFEM). The method involves FFEM for calculating fracture response characteristics; statistical models of uncertainties in load, material properties, and crack geometry; and the first-order reliability method for predicting probabilistic fracture response and reliability of cracked structures. The sensitivity of fracture parameters with respect to crack size, required for probabilistic analysis, is calculated using continuum shape sensitivity analysis. Numerical examples based on mode-I and mixed-mode problems are presented to illustrate the proposed method. The results show that the predicted failure probability based on the proposed formulation of the sensitivity of fracture parameter is accurate in comparison with the Monte Carlo simulation results. Since all gradients are calculated analytically, reliability analysis of cracks can be performed efficiently using FFEM.

1. Introduction

Fracture mechanics theory provides accurate deterministic relationship between the maximum allowable external loads and cracked structure parameters: dimensions, material properties, crack size and location. However, due to uncertainties in some of these parameters (for instance, crack size and material properties) a purely deterministic approach provides an incomplete picture of the reality. Probability theory determines how the uncertainties in crack size, loads, and material properties, when modeled accurately, affect the integrity of cracked structures. Therefore, a probabilistic approach characterizing statistical uncertainties in loads, material properties, and geometry and quantifying their impact on fracture response and integrity of materials and structures seems to be very helpful for practical engineering. Stochastic fracture mechanics (SFM), which blends the classical fracture mechanics and the probability theory, accounts for both mechanistic and statistical aspects of the crack-driving force and provides probabilistic characteristics of fracture initiation and growth of an existing crack, real or postulated, in an engineering structure [1].

To date, several methods have been developed or implemented for estimating statistics of various fracture response and reliability. Most of these methods are based on linear-elastic fracture mechanics (LEFM) and a finite element method (FEM) that employs the stress intensity factor (SIF) as the primary crack-driving force [1–6]. For example, using SIFs from an FEM code, Grigoriu et al. [2] applied first- and second-order reliability methods (FORM/SORM) to predict the probability of fracture initiation and a confidence interval of the direction of crack extension. The method can account for random loads, material properties, and crack geometry. However, the randomness in crack geometry is modeled by response surface metamodel of SIFs as explicit functions of crack geometry. Similar response surface based methods involving elastic–plastic fracture mechanics and the *J*-integral based ductile tearing theory have also appeared [7–9]. For example, a stochastic model based on an engineering approximation of the *J*-integral and FORM/SORM have been developed by Rahman and co-workers for fracture analysis of cracked tubular structures [7]. Based on this model, the probability of fracture initiation and subsequent fracture instability can be predicted under elastic–plastic conditions. The response surface approximation used in these SFM analyses significantly reduces the complexity in calculating the derivatives of the SIF or the *J*-integral. Essentially, this presents a primary rationale for successful development of FORM/SORM algorithms for probabilistic analysis of cracked structures. However, the usefulness of response surface based methods is limited, since they cannot be applied to general fracture mechanics analysis. Because of the complexity in crack geometry, external loads, and material behavior, more advanced computational tools, such as FEM, must be employed to provide the necessary computational framework for analysis of general cracked structures. Furthermore, due to various approximations in response surface metamodels, one needs to evaluate their accuracy by comparing with generally more accurate FEM based probabilistic analysis [6].

Recently, methods based on fractal geometry concepts to generate infinite number of finite elements around the crack tip to capture the crack tip singularity have been developed or investigated

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doi:10.1016/j.cma.2008.08.014

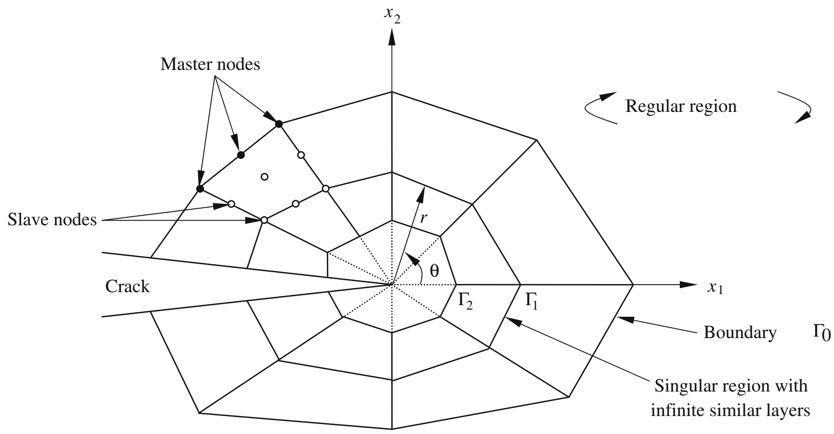

Fig. 1. Cracked body domain with regular region, singular region, and fractal mesh.

to solve LEFM problems [10–14]. The fractal finite element method (FFEM) is one such method developed for calculating the SIFs in linear-elastic crack problems. Since its origin, it has been successfully applied to solve many kinds of crack problems under mode-I and mixed mode loading conditions in 2D [15–25] and 3D [26]. Basically, FFEM separates a 2D or 3D cracked elastic body into a regular and a singular region (see Fig. 1), with the latter enclosing the crack tip. Both the regular and the singular regions are modeled by conventional, isoparametric finite elements. However, within the singular region an infinite number of elements are generated by a self-similar, fractal process to capture the singular behaviour at the crack tip. The nodal displacements in the singular region are transformed to a set of unknown coefficients using William's analytical solution for the displacements near the crack tip [27]. Since the stiffness matrix of an isoparametric element depends only on its shape and not its actual dimensions, the above transformation can be performed at the element-level and the results summed up as geometrical progression series to be assembled to the global stiffness matrix. The contributions of the infinite number of elements in the singular region are therefore fully accounted for while the number of degrees of freedom involved remains finite.

Compared with other numerical methods like FEM, FFEM has several advantages. First, by using the concept of fractal geometry, infinite finite elements are generated virtually around the crack tip, and hence the effort for data preparation can be minimized. Second, based on the eigenfunction expansion of the displacement fields [27,28], the infinite finite elements that generate virtually by fractal geometry around the crack tip are transformed in an expeditious manner. This results in reducing the computational time and the memory requirement for fracture analysis of cracked structures. Third, no special finite elements and post-processing are needed to determine the SIFs. Finally, as the analytical solution is embodied in the transformation, the accuracy of the predicted SIFs is high.

Although FEM based SFMs are well developed, research in stochastic FFEM has not been explored yet. This paper presents a FFEM for SFM analysis of linear-elastic cracked structures. The method comprises an FFEM as the deterministic kernel to calculate fracture response characteristics; statistical models of uncertainties in load, material properties, and crack geometry; and the FORM to predict probabilistic fracture response and reliability of cracked structures. The sensitivity of fracture parameters with respect to crack size, required for probabilistic analysis, is calculated using continuum shape sensitivity analysis of mixed-mode fracture in conjunction with FFEM [29]. Numerical examples based on mode-I and mixed-mode loaded cracked structures are presented to illustrate the proposed method.

2. Sensitivities of fracture parameters

2.1. Fractal finite element method

As depicted in the Fig. 1, FFEM divides the domain of a two dimensional body into a regular and a singular region, with the latter enclosing the crack tip. In the Fig. 1, the boundary curve Γ_0 separates the two regions. Both the regular and singular regions are modeled using conventional finite elements. With the crack tip as the centre of similarity and using ξ as the similarity ratio, an infinite set of curves $\{\Gamma_1, \Gamma_2, \ldots\}$, similar to Γ_0 but with proportional constants $(\xi^1, \xi^2, \ldots)$, are generated inside the singular region. Between the two curves Γ_{k-1} and Γ_k, the region is named the kth layer. Straight lines that connect the crack tip to the corner nodes lying on Γ_0 are then created, dividing each layer into a mesh of elements with a similar pattern in the process. A fractal mesh is thus generated in the singular region with conventional finite elements only being used. All nodes located on Γ_0 are called the master nodes (m), while those inside Γ_0 are called the slave nodes (s).

2.1.1. William's eigenfunction expansion

For a plane crack with traction-free faces subjected to arbitrary far-field loading, the linear-elastic displacement field at the crack tip obtained by the William's eigenfunction expansion technique [27] can be expressed as

$$u = \sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} [a_n^{\mathrm{I}} G_{11}(n,\theta) + a_n^{\mathrm{II}} G_{12}(n,\theta)], \tag{1}$$

$$v = \sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} [a_n^{\mathrm{I}} G_{21}(n,\theta) + a_n^{\mathrm{II}} G_{22}(n,\theta)], \tag{2}$$

where μ is the shear modulus and $G_{ij}(n,\theta)$, with $i; j = 1,2$, are the angular functions as given below:

$$G_{11}(n,\theta) = \left(\kappa + \frac{n}{2} + (-1)^n\right) \cos\frac{n}{2}\theta - \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta, \tag{3}$$

$$G_{12}(n,\theta) = -\left(\kappa + \frac{n}{2} - (-1)^n\right) \sin\frac{n}{2}\theta + \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta, \tag{4}$$

$$G_{21}(n,\theta) = \left(\kappa - \frac{n}{2} - (-1)^n\right) \sin\frac{n}{2}\theta + \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\theta, \tag{5}$$

$$G_{22}(n,\theta) = \left(\kappa - \frac{n}{2} + (-1)^n\right) \cos\frac{n}{2}\theta + \frac{n}{2} \cos\left(\frac{n}{2} - 2\right)\theta, \tag{6}$$

where r and θ are the polar coordinates, $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain with ν being the Poisson's ratio.

The coefficients $a_n^{\mathrm{I,II}}$ can be determined after imposing loading and other boundary conditions. Mode-I and mode-II SIFs, K_{I} and K_{II} are related to the first degree coefficients $(a_1^{\mathrm{I,II}})$ in the series which are directly associated with the $r^{-1/2}$ term in the stresses