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Spectral element FETI-DP and BDDC preconditioners with multi-element subdomains

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ABSTRACT

The discrete systems generated by spectral or *hp*-version finite elements are much more ill-conditioned than the ones generated by standard low-order finite elements or finite differences. This paper focuses on spectral elements based on Gauss–Lobatto–Legendre (GLL) quadrature and the construction of primal and dual non-overlapping domain decomposition methods belonging to the family of Balancing Domain Decomposition methods by Constraints (BDDC) and Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) algorithms. New results are presented for the spectral multi-element case and also for inexact FETI-DP methods for spectral elements in the plane. Theoretical convergence estimates show that these methods have a convergence rate independent of the number of subdomains and coefficient jumps of the elliptic operator, while there is only a polylogarithmic dependence on the spectral degree p and the ratio H/h of subdomain and element sizes. Parallel numerical experiments on a Linux cluster confirm these results for tests with spectral degree up to $p = 32$, thousands of subdomains and coefficient jumps up to 8 orders of magnitude.

1. Introduction

High-order finite element methods based on spectral elements or *hp*-version finite elements improve the accuracy of the discrete solution by increasing the polynomial degree of the basis functions as well as the number of elements. Usually, *hp*-version finite elements are based on hierarchical non-nodal basis functions and have been studied mostly in the structural mechanics community; see, e.g., Szabó and Babuška [54]. Spectral elements are based on tensorial nodal bases associated with Gauss–Lobatto–Legendre (GLL) quadrature nodes and have been studied mostly in the fluid dynamics community; see, e.g., Canuto et al. [11], Bernardi and Maday [6], Funaro [26], Karniadakis and Sherwin [31], Deville et al. [14]. The discrete systems generated by these high-order methods are much more ill-conditioned than the ones generated by standard low-order finite elements. While still proportional to the inverse of the square of the element size, the condition number of these high-order discrete systems is typically proportional to the cube or fourth power of the polynomial degree of the basis functions employed. Therefore, the construction of efficient preconditioners for these methods is a challenging and important issue. Particularly open to research are preconditioners for non-tensorial spectral and *hp*-finite elements, usually on triangular and tetrahedral elements; see Ainsworth [1], Bica [8], Pavarino and Warburton [47], Sherwin and Casarin [52], Giraldo and Warburton [27], Pavarino et al. [44], Schöberl et al. [51]. In this paper, we will focus on tensorial GLL spectral elements only.

Domain decomposition methods are preconditioned iterative algorithms for the solution of the large systems obtained from the discretization of partial differential equations. In domain decomposition methods, the domain associated with the partial differential equation is decomposed into a, possibly large, number of subdomains. On these subdomains, local problems are defined which are solved in each iteration step in order to define an approximate inverse of the system matrix. In order to obtain a numerical and parallel scalable algorithm, also a small coarse problem has to be introduced and solved in each iteration step.

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In this article, we consider non-overlapping domain decomposition methods belonging to the family of Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) methods, see Farhat et al. [21,22], Klawonn et al. [36,33,35,38], and their primal counterparts, the algorithms known as Balancing Domain Decomposition methods by Constraints (BDDC); see Dohrmann [15], Mandel and Dohrmann [41], Mandel, Dohrmann, and Tezaur [42], Cros [13], or Li and Widlund [39]. In these methods, the condition number only depends weakly on the polynomial degree. We will also consider inexact versions of the FETI-DP methods; see Klawonn and Rheinbach [34]. Inexact BDDC methods have been considered by Tu [57,56] and recently by Li and Widlund [40] and Dohrmann [16]. The family of FETI-DP algorithms descended from the earlier one-level and two-level FETI algorithms, see Farhat and Roux [25,24], Farhat, Mandel, and Roux [23], Farhat and Mandel [19], Farhat, Pierson, and Lesoinne [20]. Already the FETI method has been used in large-scale parallel simulations, e.g. [7].

We note that preliminary serial results on a smaller scale for FETI-DP and BDDC spectral element preconditioners with only one element for each subdomain and only exact solvers have been considered by Pavarino [46]. Also, in [53] FETI and FETI-DP methods for spectral elements were already compared for the case of a single spectral element per subdomain. In this paper, we present new results for both BDDC and FETI-DP methods for spectral element discretizations with multi-element subdomains, efficient inexact coarse solvers for FETI-DP and a scalable parallel implementation. These results allow us to extend the previous serial results of [53,46] to large-scale tests with spectral degree up to $p = 32$, thousands of subdomains and coefficient jumps up to 8 orders of magnitude, thus confirming the theoretical bound on the condition number of the preconditioned operators.

2. Spectral element discretization of second order elliptic problems

Let T_{ref} be the reference square $(-1,1)^2$, and let $Q_{\text{p}}(T_{\text{ref}})$ be the set of polynomials on T_{ref} of degree $p \geqslant 1$ in each variable. We assume that the domain Ω can be decomposed into N_e non-overlapping finite elements T_k of characteristic diameter h

$$\overline{\Omega} = \bigcup_{k=1}^{N_e} \overline{T}_k, \tag{1}$$

each of which is an affine image of the reference square, $T_k = \phi_k(T_{\text{ref}})$, where ϕ_k is an affine mapping (more general maps could be considered as well). In the next section, we will group these elements into N non-overlapping subdomains Ω_i of characteristic diameter H, forming themselves a coarse finite element partition of Ω

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i, \quad \overline{\Omega}_i = \bigcup_{k=1}^{N_i} \overline{T}_k. \tag{2}$$

Hence the fine element partition (1) can be considered a refinement of the coarse subdomain partition $\{\Omega_i\}_{i=1}^N$ in (2), with matching finite element nodes on the boundaries of neighboring subdomains.

We consider linear, selfadjoint, elliptic problems on Ω, with zero Dirichlet boundary conditions on a part $\partial\Omega_D$ of the boundary $\partial\Omega$:

Find $u \in V = \{v \in H^1(\Omega) : \; v = 0 \text{ on } \partial\Omega_D\}$ such that

$$a(u,v) = \int_\Omega \rho(x)\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V. \tag{3}$$

Here $\rho(x) > 0$ can be discontinuous, with very different values for different subdomains, but we assume this coefficient to vary only moderately within each subdomain Ω_i. In fact, without decreasing the generality of our results, we will only consider the piecewise constant case of $\rho(x) = \rho_i$, for $x \in \Omega_i$.

Conforming spectral element discretizations consist of continuous, piecewise polynomials of degree p in each element:

$$V^p = \{v \in V : v|_{T_i} \circ \phi_i \in Q_p(T_{\text{ref}}), \; i = 1, \dots, N_e\}. \tag{4}$$

A convenient tensor product basis for V^p is constructed using Gauss–Lobatto–Legendre (GLL) quadrature points; other bases could be considered, such as those based on integrated Legendre polynomials common in the p-version finite element literature; see Szabó and Babuška [54]. Let $\{\xi_i\}_{i=0}^p$ denote the set of GLL points on $[-1,1]$, and let σ_i denote the quadrature weight associated with ξ_i. Let $l_i(\cdot)$ be the Lagrange interpolating polynomial which vanishes at all the GLL nodes except ξ_i, where it equals one. The basis functions, e.g., on the reference square, are then defined by a tensor product as

$$l_i(x_1)l_j(x_2), \quad 0 \leqslant i, \quad j \leqslant p.$$

This basis is nodal, since every element of $Q_{\text{p}}(T_{\text{ref}})$ can be written as

$$u(x_1,x_2) = \sum_{i=0}^p \sum_{j=0}^p u(\xi_i,\xi_j) l_i(x_1) l_j(x_2).$$

Each integral of the continuous model (3) is replaced by GLL quadrature. On T_{ref}

$$(u,v)_{p,T_{\text{ref}}} = \sum_{i=0}^p \sum_{j=0}^p u(\xi_i,\xi_j) v(\xi_i,\xi_j) \sigma_i \sigma_j,$$

and on all of Ω

$$(u,v)_{p,\Omega} = \sum_{k=1}^{N_e} \sum_{i,j=0}^{p} (u \circ \phi_k)(\xi_i,\xi_j)(v \circ \phi_k)(\xi_i,\xi_j)|J_k|\sigma_i\sigma_j,$$

where $|J_k|$ is the determinant of the Jacobian of ϕ_k. This inner product is uniformly equivalent to the standard L_2-inner product on $Q_{\text{p}}(T_{\text{ref}})$:

$$\|u\|^2_{L_2(T_{\text{ref}})} \leqslant (u,u)_{p,T_{\text{ref}}} \leqslant 27\|u\|^2_{L_2(T_{\text{ref}})} \quad \forall u \in Q_{\text{p}}(T_{\text{ref}}), \tag{5}$$

see Bernardi and Maday [6]. These bounds imply an analogous uniform equivalence between the $H^1(\Omega)$-seminorm and the discrete seminorm $(\nabla u, \nabla u)_{n,\Omega}$ based on GLL quadrature. Applying these quadrature rules, we obtain the discrete bilinear form

$$a_{\text{p}}(u,v) = \sum_{k=1}^{N_e} (\rho_k \nabla u, \nabla v)_{p,T_k},$$

and the discrete elliptic problem:

Find $u \in V^p$ such that

$$a_{\text{p}}(u,v) = (f,v)_{\text{p},\Omega} \quad \forall v \in V^p. \tag{6}$$

Having chosen a basis for V^p, the discrete problem (6) is then turned into a linear system of algebraic equations

$$K_{\text{g}} u_{\text{g}} = f_{\text{g}}, \tag{7}$$

where K_{g} is the globally assembled, symmetric, positive definite stiffness matrix. We have denoted with the same symbols u_{g} and f_{g} the vectors representing the corresponding spectral element functions in the given basis.

3. FETI-DP and BDDC domain decomposition

3.1. Primal and dual iterative substructuring

In substructuring methods, the computational domain $\Omega \subset \mathbf{R}^2$ is decomposed into non-overlapping subdomains Ω_i. Let us denote by N the number of subdomains, each subdomain is the union of finite