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Analytic modeling of laminar forced convection in a circular duct for arbitrary boundary conditions and inlet temperature profile

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ABSTRACT

Although steady forced laminar convection in a circular duct is a classical problem in heat transfer, analytical solutions are only available in a quite limited set of boundary conditions. In this work, a general analytic solution is derived, which is valid for arbitrary wall boundary conditions as well as arbitrary inlet temperature profiles. Solution covers thermally developing as well as thermally fully developed domains, but is limited to hydrodynamically fully developed flow. As a result, the concept of thermally fully developed flow, hence thermal entrance length, had to be revisited. A new general definition is proposed here, which englobes classical definition and generalizes it to arbitrary boundary conditions. The new entrance length concept, which is of fundamental nature, is introduced for a rather general duct cross-section, although remaining analysis to obtain temperature field is presented for the circular cross-section only.

1. Motivation

Problem considered here is that of steady laminar heat transfer by forced convection in a straight circular duct. Surprisingly, although this problem is quite common in engineering practice, it has only been considered for a limited, as well as unrealistic, set of 'standard' cases. Abundant literature is available for different boundary conditions and duct cross sections. Nevertheless, of all types of duct wall boundary conditions, only two cases were considered in the literature:

– Uniform duct wall heat flux density q_w
– Uniform duct wall temperature T_w

Moreover, among all possible fluid inlet temperature T_{in} profiles, only two cases were considered:

– Uniform (also called flat)
– Fully developed, i.e. T_{in} is the same as outlet profile for a sufficiently long duct

Unfortunately, each of the above conditions is very difficult to realize in practice. Over a century ago, Graetz [1] have presented the solution of steady forced laminar convection in a circular duct due to a flat non-zero inlet temperature profile and uniform wall temperature ($T_w = 0$). Axial heat transfer by conduction was neglected in this, so-called, 'standard' Graetz problem. Over the following years, many variants were proposed to improve it. Taking axial conduction into consideration was a common factor in what is commonly called 'extended' Graetz problem. Shah & London [2] have presented a comprehensive work for many cross sections, circular and non-circular. Papoutsakis et al. [3] presented an analytical solution for a uniform heat flux over a segment of duct wall, together with a uniform temperature far upstream of the heated section. This is a quite interesting approach, but it does not address arbitrary inlet temperature profiles. In real problems, due to the finiteness of wall thermal conductivity, wall heat flux density is never discontinuous. Based on [3], Colle [4] has partially solved this problem by extending the solution to cover what he called 'arbitrary' boundary conditions. However, it is still limited to cases where temperature asymptotically tends to uniform values both upstream and downstream. It represents part of the more general solution proposed here, which is precisely the thermal entrance region, but not the thermal fully developed region. Lawal & Mujumdar [5] have considered internal heat generation for a uniform wall temperature and flat inlet profile. Ebadian & Zhang [6] have considered the effect of a finite jump in wall temperature between two uniform but different temperatures. Bilir [7] numerically studied heat transfer for both uniform wall temperature and uniform wall heat flux. Wiegand et al. [8] considered piecewise uniform heat flux. Combined effect of wall conduction was studied by Maranzana et al. [9] and Wiegand & Gassner [10]. De Barros & Sphair [11] studied. Del Giudice et al. [12] and Belhocine [13,12] studied. Shamardan et al. have studied the fully developed case

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with uniform wall heat flux for rectangular [14] and triangular [15] duct cross sections.

In case we need to model a realistic wall and/or inlet conditions, only two choices are offered classically:

- Either continue using results of any of the standard cases mentioned above, as an approximation that can be crude, or
- Numerically solve heat transfer partial differential equations together with realistic boundary conditions as a more precise but cumbersome procedure.

The purpose of this work is to ***construct a full and general model*** of steady forced convection in a straight circular duct. The model will be able to treat ***arbitrary*** variable duct wall boundary conditions, not just the two cases enumerated above. The model will also be able to treat an ***arbitrary*** variable inlet temperature profile, not just the two cases enumerated above. For arbitrary, hence non-uniform, boundary conditions, the classical entrance length concept becomes meaningless. This work presents another contribution of fundamental character in Section 3, which is a generalization of the thermal entrance length concept that is valid for both uniform and non-uniform boundary conditions.

The model will yield (Section 4) full analytical solution giving temperature profile at all cross sections from inlet to exit, including variation of wall temperature T_w for any given variation of duct wall heat flux density q_w. This can be used to replace the well-known engineering relation between q_w and T_w, which is usually cast in the following over simplified form (the so-called Newton law of cooling):

$$q_w = h(T_w - T_b) \tag{1}$$

where h is the so-called heat transfer coefficient HTC and T_b is the bulk fluid temperature. Correlations are available for h for only the standard cases cited above. Expression (1) is usually justified by the linearity of equations governing forced convection. In fact, assuming uniform physical properties and negligible heat dissipation by internal friction, heat equation is:

$$\rho c \mathbf{v} \cdot \nabla T = k \nabla^2 T \tag{2}$$

In above equation, ρ is the density, c heat capacity, $\mathbf{v}$ fluid velocity, T temperature and k thermal conductivity. For forced convection and constant fluid properties, $\mathbf{v}$ is independent of T. Hence, the relation between temperature and heat flux should be linear.

The problem is that (1) is ***not the only*** possible linear relation. It has been proved in [16,17] that the ***most general*** linear relation between q and T has the form:

$$T(\mathbf{x}) - T_{ref} = \int G(\mathbf{x}, \mathbf{x}') q(\mathbf{x}') d\mathbf{x}' \tag{3}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are position vectors, $G(\mathbf{x}, \mathbf{x}')$ the Green's function of the original partial differential Eq. (2) and T_{ref} is a reference temperature that depends on the derivation of G. It was also shown in [11,16] that the assumptions required to transform (3) into (1) may introduce large errors, which explains in part the disparity observed between published HTC correlations. Integration in (3) is performed over any zone where heat may be exchanged with the surroundings (wall or bulk). Resulting temperature is valid anywhere in the whole domain, not just the wall. Obtaining Green's function analytically is possible for simple geometries. For arbitrary geometries, an approximate procedure has also been proposed in [16]. The present paper can be considered as an application of these general concepts to forced convection in a circular duct, where analytical solutions can be derived, as will be made in the present work.

2. Problem description

For simplicity, problem considered is hydrodynamically fully developed, with a given velocity profile that is the same over any cross section normal to fluid flow. Let us designate the coordinate along the straight circular duct (Fig. 1) by z (ranging between 0 at inlet and duct length L at outlet), and the velocity along it by w. Eq. (2) becomes:

$$\rho c w \partial T / \partial z = k(\nabla_2^2 T + \partial^2 T / \partial z^2) \tag{4}$$

Here ∇_2^2 designates the 2D Laplacian over duct cross section. Problem domain Ω, has the following set of boundaries: duct wall $\partial\Omega_w$, inlet cross section $\partial\Omega_{in}$ and outlet cross section $\partial\Omega_{out}$. Boundary conditions are specified as follows:

$$q|_{\partial\Omega_w} = k(\mathbf{n} \cdot \nabla T)|_{\partial\Omega_w} = q_w; \quad T|_{\partial\Omega_{in}} = T_{in} \tag{5}$$

where $\mathbf{n}$ is the unit outward normal, while q_w is the entering wall heat flux density. No boundary conditions will be imposed on the outlet, as usually done, assuming flow velocity is sufficient to let the influence of this condition be limited to a very narrow zone near the outlet.

Wall boundary condition is still a general one in all respects. First, wall heat flux profile q_w is arbitrary. Second, the type of boundary condition selected (i.e. Neumann) is not a restriction either. In fact, the analytical solution to be obtained, will give us temperature everywhere, including at the wall, as a function of the arbitrary heat flux imposed. For Dirichlet (or Robin) boundary conditions, i.e. given T_w or a relation between T_w and q_w, a simple procedure can be followed to retrieve back corresponding q_w compatible with given data.

For convenience, the bulk temperature at inlet T_{b_in} will be taken as the reference temperature. Equations will be cast in a non-dimensional form using as a unit length R (half of the hydraulic diameter D_h) and a characteristic temperature difference ΔT_{ref} defined as $R\, q_{avg}/k$ (where q_{avg} is a characteristic value of heat flux density). Finally, velocity field will be normalized by the average discharge velocity v_{avg}. New dimensionless variables are:

$$z' = z/R; T' = T/\Delta T_{ref}; q'_w = q_w/q_{avg}; \quad w' = w/v_{avg} \tag{6}$$

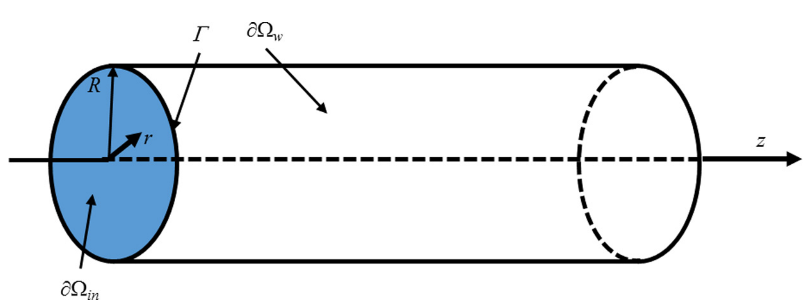

Fig. 1. Problem domain.