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The finite cell method for three-dimensional problems of solid mechanics

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ABSTRACT

This article presents a generalization of the recently proposed finite cell method to three-dimensional problems of linear elasticity. The finite cell method combines ideas from embedding or fictitious domain methods with the *p*-version of the finite element method. Besides supporting a fast, simple generation of meshes it also provides high convergence rates. Mesh generation for a boundary representation of solids and for voxel-based data obtained from CT scans is addressed in detail. In addition, the implementation of non-homogeneous Neumann boundary conditions and the computation of cell matrices based on a composed integration is presented. The performance of the proposed method is demonstrated by three numerical examples, including the elastostatic computation of a human bone biopsy.

1. Introduction

The finite cell method (FCM), which was recently proposed by the authors [1] can be interpreted as a combination of a fictitious or embedding domain approach with high-order finite element methods. It therefore combines the fast, simple generation of meshes with high convergence rates. In fictitious domain methods the original or physical domain is embedded in a geometrically larger domain of a simpler shape. Thanks to the simple geometry of the embedding domain it can be readily discretized with structured or Cartesian grids. Different discretization methods can be applied, ranging from finite difference to finite volume and finite element methods. The name "fictitious domain method" was coined by Saul'ev [2,3] in the early sixties. Since then the fictitious domain method has been further developed and applied to model problems arising in different areas of computational mechanics. For an overview of the huge body of literature, please refer to [4–6].

Perhaps the most relevant work on fictitious domain methods with regard to the current paper was published by Bishop [7] and Ramière et al. [5]. Bishop has used an implicit meshing for two-dimensional problems of linear elasticity to discretize the embedding domain. An algorithm to integrate the weak form exactly is suggested in order to account for elements which are cut by the physical boundary. It is shown that bi-cubic Hermite elements yield more accurate and efficient results of displacements and stresses than bi-quadratic Lagrange elements. This could provide a clue for increasing the order of approximation space for better results. Increase in accuracy from quadratic Lagrange elements, that are C^0-continuous, to Hermite cubics, that are C^1-continuous, may also be attributed to the increase in the level of continuity or smoothness of the underlying discretization. For the role of continuity in the discretization of solids and fluids in the context of isogeometric analysis the reader is referred to [8–10]. Despite interesting achievements of Bishop [7], the algorithm to integrate the weak form exactly is likely to become very expensive for three-dimensional problems. Since the volume integrals are converted to boundary integrals by means of the divergence theorem, the approach is restricted to element-wise constant data, which limits the approach to problems with homogeneous and linear material or precludes the application of high-order shape functions for nonlinear and inhomogeneous materials.

For Ramière et al., the core idea is again to immerse the original domain into a simpler, geometrically larger one. Both finite volume and bi-linear finite element methods are used to solve elliptic problems with general boundary conditions. Different methods for treating the boundary conditions are discussed. The literature provides a wide scope of ideas and techniques for imposing boundary conditions which are similar in nature but go by different names. For a review of such techniques the reader is referred to [4].

In the finite cell method, as proposed by the authors [1], the idea is to use an easily discretized domain in which the physical domain is embedded. Therefore, as in all similar methods that

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are not based on a boundary-conforming mesh, the accuracy in discretizing the domain is replaced by an accurate integration scheme. Assuming a soft material which fills the void regions of the embedding domain makes a standard finite element discretization possible. However, the fast convergence to an accurate result is due to the fact that a high-order Ansatz space is used [11,12]. A dense distribution of integration points serves both to capture the boundary, as in level sets [13], and to increase the accuracy of integration over cells that are independent of the physical domain.

The structure of the paper is as follows: Section 2 summarizes the finite cell formulation for three-dimensional problems of linear elastostatics. This includes the variational formulation, the implementation of boundary conditions and the computation of cell matrices. Section 3 discusses the generation of meshes for different types of geometric models. The performance of the proposed method is presented in Section 4 using three numerical examples.

2. The finite cell method

We closely follow the description in [1] to explain the finite cell method. For better clarity and understanding, the figures are presented in two dimensions but the formulation applies similarly in three dimensions, as demonstrated in the examples.

2.1. Variational formulation

Let us assume that on a three-dimensional physical domain Ω, a problem of linear elasticity is described by the weak form of equilibrium as

$$\mathscr{B}(\mathbf{u},\mathbf{v}) = \mathscr{F}(\mathbf{v}), \tag{1}$$

where the bilinear part is

$$\mathscr{B}(\mathbf{u},\mathbf{v}) = \int_{\Omega} [\mathbf{L}\mathbf{v}]^{\mathrm{T}}\mathbf{C}[\mathbf{L}\mathbf{u}]\,\mathrm{d}\Omega, \tag{2}$$

in which $\mathbf{u}$ is the displacement, $\mathbf{v}$ is the test function, $\mathbf{L}$ is the standard strain–displacement operator and $\mathbf{C}$ is the elasticity matrix. Note that the principles of the method are not restricted to linear differential operators. Without loss of generality we assume homogeneous Dirichlet boundary conditions $\bar{\mathbf{u}} = \mathbf{0}$ along Γ_{D} and a Neumann boundary Γ_{N} with prescribed tractions, $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, and $\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{N}} = \emptyset$. The linear functional

$$\mathscr{F}(\mathbf{v}) = \int_{\Omega} \mathbf{v}^{\mathrm{T}}\mathbf{f}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{v}^{\mathrm{T}}\bar{\mathbf{t}}\,\mathrm{d}\Gamma \tag{3}$$

takes the volume loads $\mathbf{f}$ and prescribed tractions $\bar{\mathbf{t}}$ into account.

The original physical domain can now be embedded in the domain Ω_{e} with the boundary $\partial\Omega_{\mathrm{e}}$. For the sake of simplicity, the situation is depicted for a two-dimensional case in Fig. 1. The interface between Ω and the embedding domain is defined as $\Gamma_{\mathrm{I}} = \partial\Omega \backslash (\partial\Omega \cap \partial\Omega_{\mathrm{e}})$. Following Neittaanmäki and Tiba [14], the displacement variable is extended as

$$\mathbf{u} = \begin{cases} \mathbf{u}^1 & \text{in } \Omega, \\ \mathbf{u}^2 & \text{in } \Omega_{\mathrm{e}} \setminus \Omega, \end{cases} \tag{4}$$

while the transition conditions guarantee continuity at the interface between Ω and $\Omega_{\mathrm{e}} \backslash \Omega$:

$$\begin{aligned} \mathbf{u}^1 &= \mathbf{u}^2 \quad \text{on } \Gamma_I, \\ \mathbf{t}^1 &= \mathbf{t}^2 \quad \text{on } \Gamma_I. \end{aligned} \tag{5}$$

Boundary conditions are set for $\partial\Omega_{\mathrm{e}}$

$$\begin{aligned} \bar{\mathbf{u}} &= \mathbf{0} \quad \text{on } \Gamma_{\mathrm{e,D}}, \\ \bar{\mathbf{t}} &= \mathbf{0} \quad \text{on } \Gamma_{\mathrm{e,N}}, \end{aligned} \tag{6}$$

in which $\Gamma_{\mathrm{e,D}}$ and $\Gamma_{\mathrm{e,N}}$ are the Dirichlet and Neumann boundaries of Ω_{e} respectively, $\partial\Omega_{\mathrm{e}} = \Gamma_{\mathrm{e,D}} \cup \Gamma_{\mathrm{e,N}}$, and $\Gamma_{\mathrm{e,D}} \cap \Gamma_{\mathrm{e,N}} = \emptyset$. The first condition in (6) is generally necessary to avoid rigid body motion. The weak form of the equilibrium equation for the embedding domain Ω_{e} is given as

$$\mathscr{B}_{\mathrm{e}}(\mathbf{u},\mathbf{v}) = \mathscr{F}_{\mathrm{e}}(\mathbf{v}), \tag{7}$$

where the bilinear form is

$$\mathscr{B}_{\mathrm{e}}(\mathbf{u},\mathbf{v}) = \int_{\Omega_{\mathrm{e}}} [\mathbf{L}\mathbf{v}]^{\mathrm{T}}\mathbf{C}_{\mathrm{e}}\ [\mathbf{L}\mathbf{u}]\,\mathrm{d}\Omega, \tag{8}$$

in which

$$\mathbf{C}_{\mathrm{e}} = \alpha\mathbf{C} \tag{9}$$

is the elasticity matrix of the embedding domain, with

$$\alpha(\mathbf{x}) = \begin{cases} 1.0 & \forall \mathbf{x} \in \Omega, \\ 0.0 & \forall \mathbf{x} \in \Omega_{\mathrm{e}} \setminus \Omega. \end{cases} \tag{10}$$

Inserting (9) and (10) into (8), the bilinear functional turns to

$$\begin{aligned} \mathscr{B}_{\mathrm{e}}(\mathbf{u},\mathbf{v}) &= \int_{\Omega_{\mathrm{e}}} [\mathbf{L}\ \mathbf{v}]^{\mathrm{T}}\alpha\mathbf{C}[\mathbf{L}\mathbf{u}]\,\mathrm{d}\Omega \\ &= \int_{\Omega} [\mathbf{L}\mathbf{v}]^{\mathrm{T}}\mathbf{C}[\mathbf{L}\mathbf{u}]\,\mathrm{d}\Omega + \int_{\Omega_{\mathrm{e}}\backslash\Omega} [\mathbf{L}\mathbf{v}]^{\mathrm{T}}\mathbf{0}[\mathbf{L}\mathbf{u}]\,\mathrm{d}\Omega = \mathscr{B}(\mathbf{u},\mathbf{v}). \end{aligned} \tag{11}$$

The linear functional

$$\mathscr{F}_{\mathrm{e}}(\mathbf{v}) = \int_{\Omega_{\mathrm{e}}} \mathbf{v}^{\mathrm{T}}\alpha\mathbf{f}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{v}^{\mathrm{T}}\bar{\mathbf{t}}\,\mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{e,N}}} \mathbf{v}^{\mathrm{T}}\bar{\mathbf{t}}\,\mathrm{d}\Gamma \tag{12}$$

considers the volume loads $\mathbf{f}$, prescribed traction along Γ_{N} *interior* to Ω_{e} and prescribed traction at the boundary of the embedding domain. Due to Eq. $(6)_2$, the last term in (12) can be assumed 0.

The embedding domain is now discretized in a mesh which is independent of the original domain. These “finite elements” of the embedding domain do not necessarily fulfill the usual geometric properties of elements for the original domain Ω, as they may be intersected by $\partial\Omega$. To distinguish them from classical elements they will be called *finite cells*. It is simpler and more advantageous to initially assume cells to be rectangular hexahedrals (cuboids) resulting in a constant Jacobian matrix of the cell-wise mapping. Fig. 2 illustrates the situation for a two-dimensional setting. The union of all n_{c} cells forms the embedding domain

$$\Omega_{\mathrm{e}} = \bigcup_{c=1}^{n_{\mathrm{c}}} \Omega_{\mathrm{c}}, \tag{13}$$

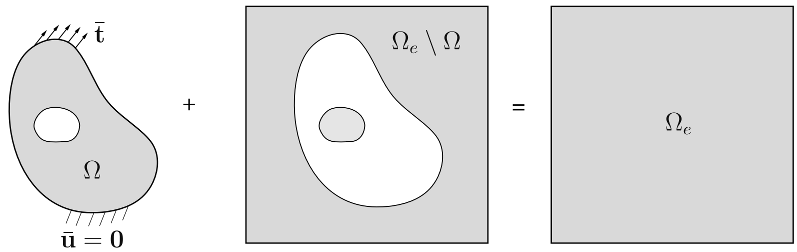

Fig. 1. The domain Ω is embedded in Ω_{e}.