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Solving inverse coefficient problems of non-uniform fractionally diffusive reactive material by a boundary functional method

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ABSTRACT

The inverse coefficient problems for estimating the unknown space-time dependent conductivity function and reaction coefficient function of a time-fractional diffusion reaction equation for non-uniform material are solved in the paper, without needing of initial condition, final time condition and internal measurement. We tackle these inverse coefficient problems supplementing with boundary data. After a homogenization technique by using these boundary data, a sequence of boundary functions are derived, which together with the zero element constitute a linear space. As an approximation, a boundary functional is proved in the linear space, of which the time-dependent energy is preserved for the energetic boundary function at each time step. The linear systems and iterative algorithms used to recover the unknown conductivity function and reaction coefficient with energetic boundary functions as bases are developed, which are convergent fast at each time step. The data required are merely the boundary temperatures and fluxes, and the boundary values and slopes of unknown functions to be recovered. The accuracy and robustness of present methods are confirmed by comparing the estimated results under large noise with exact solutions.

1. Introduction

The time fractional diffusion equation can be used to model superdiffusion and subdiffusion phenomena [1–4]. It has been studied as a complicated dynamical system [5–7], and had extensive applications in many fields such as chemotaxis model [8], and porous and fractured media [9,10]. The anomalous diffusion is different from the normal diffusion in that while in the normal diffusion, particle is in a Brownian motion and the mean square displacement is a linear function of time, the anomalous diffusion is essentially one kind of non-locality non-Markovian motion, such that the time-space relativity must be taken into account and at the same time the particle motion is not of the Brownian type, and the mean square displacement is not a linear function of time.

Many works on the inverse source problems of fractional diffusive equation have been carried out, for which the final time data or an internal measurement of temperature is required, and due to their high ill-posedness in nature, a regularization is required [11–16].

In contrast, there are only a few works to identify the coefficients $a(x)$ and $p(x)$ in the following time-fractional diffusive reactive equation [17–21]:

$$D_t^\alpha u(x,t) = \frac{\partial[a(x)u_x(x,t)]}{\partial x} + p(x)u(x,t) + H(x,t), \tag{1}$$

where the subscript x denotes the differential with respect to x, and $H(x,t)$ is a given source function. The time-fractional derivative $D_t^\alpha u(x,t)$ is defined in the Caputo sense [22–24]:

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds, 0<\alpha<1, \tag{2}$$

where $u_s(x,s)$ denotes the partial differential of $u(x,s)$ with respect to s. When $\alpha = 1$ we recover to the usual first-order time differential.

We consider an inverse coefficient problem to recover an unknown conductivity function $a(x,t)$ in a one-dimensional time-fractional diffusion equation, of which one needs to find the temperature distribution $u(x,t)$ as well as the conductivity function $a(x,t)$ that simultaneously satisfy

$$D_t^\alpha u(x,t) = \frac{\partial[a(x,t)u_x(x,t)]}{\partial x} + H(x,t), \quad 0<x<\ell, \quad 0<t\leqslant t_f. \tag{3}$$

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Nomenclature

$a(x,t)$	conductivity coefficient
$a_0(t)$	$:= a(0,t)$
$a_\ell(t)$	$:= a(\ell,t)$
$a'_0(t)$	$:= a_x(0,t)$
$a'_\ell(t)$	$:= a_x(\ell,t)$
b_0, b_1, b_2	coefficients defined in Eqs. (16) and (32)
$B_j(x)$	boundary functions
$B_0(x,t)$	homogenized function
c_i	expansion coefficients
D_t^α	time-fractional derivative
$d(x,t)$	function defined in Eq. (22)
$d(x,t)$	function defined in Eq. (35)
$E_j(x)$	dynamic energetic boundary functions
$F(t)$	function defined in Eq. (6)
$F_0(t)$	left boundary condition
$F_\ell(t)$	right boundary condition
$H(x,t)$	source function
ℓ	length of space
m	the highest order of energetic boundary function
$p(x,t)$	reaction coefficient
$p_0(t)$	$:= p(0,t)$
$p_\ell(t)$	$:= p(\ell,t)$
$p'_0(t)$	$:= p_x(0,t)$
$p'_\ell(t)$	$:= p_x(\ell,t)$
$Q_0(t)$	left boundary flux
$Q_\ell(t)$	right boundary flux
$R(i)$	random number
s	noise level
t	time
t_f	final time
$u(x,t)$	temperature
$v(x,t)$	$:= u(x,t) - B_0(x,t)$
x	space variable

Greek symbols

α	order of fractional derivative
γ_j	multiplier
ε	convergence criterion

Subscripts and superscripts

i	index
j	index
k	index

Because the above problem has an unknown function $a(x,t)$ and gives no initial condition, it cannot be solved directly to find $u(x,t)$. We do not need to know the initial temperature, final time temperature and internal data of $u(x,t)$ for this inverse coefficient problem; however, in order to recover $a(x,t)$ we give over-specified boundary data of

$$u(0,t) = F_0(t), \quad u(\ell,t) = F_\ell(t),$$
$$u_x(0,t) = Q_0(t), \quad u_x(\ell,t) = Q_\ell(t), \tag{4}$$
$$a(0,t) = a_0(t), \quad a(\ell,t) = a_\ell(t),$$
$$a_x(0,t) = a'_0(t), \quad a_x(\ell,t) = a'_\ell(t). \tag{5}$$

There are only a few works on the fractional inverse coefficient problem to identify $a(x,t)$, which is more difficult than that identifying $a(t)$ in [25]. In this paper we are going to propose a different approach by using the boundary measurements of $u(x,t)$ and $a(x,t)$ to recover the conductivity function $a(x,t)$. This identification technique would be much data saving and time saving in the solution of the inverse fractional identification problem of a time-fractional diffusion equation.

By using Eqs. (3)–(5) to recover the unknown conductivity function $a(x,t)$ is a very difficult task, because the system (3)–(5) is seriously under-determined, and the resulting inverse coefficient problem is severely ill-posed. For this inverse conductivity problem the measurement error may lead to a large discrepancy from the true value of $a(x,t)$. After identifying $a(x,t)$ in the first part of the paper, we will also identify $p(x,t)$ in Eq. (1) with $a(x) = 1$ and $p(x)$ replaced by $p(x,t)$. The identification of $p(x,t)$ is more difficult than that identifying $p(x)$ in [21,26,27].

The present study aims to solve the inverse coefficient problems of a time-fractional diffusion reaction equation without needing of initial temperature, final time temperature and internal measurements of temperature data. This identification technique is novel, which can find a wide range advanced applications in engineering and science, because it is often easier to measure the boundary temperatures and fluxes at a certain time, rather than to directly measure the temperatures inside the material. Due to its importance on the knowledge of non-homogeneous conductivity function and reaction coefficient function for fractional diffusion reaction equation used in many system analyses, these inverse coefficient problems have attracted a lot of attentions.

The remainder of this paper is arranged as follows. In Section 2 we introduce a new concept of dynamic energy functional in terms of boundary functions, which together with the zero element constitute a linear space of all polynomial functions with at least the fourth-order, and satisfy the homogeneous boundary conditions. In Section 3 we derive the iterative algorithm to recover the unknown conductivity function $a(x,t)$, while the numerical examples to recover $a(x)$ and $a(x,t)$ are given in Section 4. In Section 5 we derive the iterative algorithm to recover the unknown reaction coefficient function $p(x,t)$, and the numerical examples to recover $p(x)$ and $p(x,t)$ are given in Section 6. Finally, the conclusions are drawn in Section 7.

2. Energetic boundary functional method

Multiplying both the sides of Eq. (3) by $u(x,t)$, we begin with

$$u(x,t)D_t^\alpha u(x,t) = u(x,t)\partial_x[a(x,t)u_x(x,t)] + H(x,t)u(x,t).$$

Integrating it from $x = 0$ to $x = \ell$ and using integration by parts to the second term, we can derive

$$\int_0^\ell u(x,t)D_t^\alpha u(x,t)dx + \int_0^\ell a(x,t)u_x^2(x,t)dx - \int_0^\ell H(x,t)u(x,t)dx = a_\ell(t)Q_\ell(t)F_\ell(t) - a_0(t)Q_0(t)F_0(t) =: F(t), \tag{6}$$

where the boundary conditions (4) and (5) were used, such that $F(t)$ is a given function of time. The resulting equation is a dynamic energy equation due to its time-dependence, which motivates us to use the dynamic energy functional as a mathematical tool to identify $a(x,t)$.

Usually, the given data in Eq. (4) are not zero, which leaves an obstacle to set up a linear space to be introduced below. Before embarking the analysis we seek a variable transformation by

$$v(x,t) = u(x,t) - B_0(x,t), \tag{7}$$