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A novel alpha finite element method (αFEM) for exact solution to mechanics problems using triangular and tetrahedral elements

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A B S T R A C T

The paper presents an alpha finite element method (αFEM) for computing nearly exact solution in energy norm for mechanics problems using meshes that can be generated automatically for arbitrarily complicated domains. Three-node triangular (αFEM-T3) and four-node tetrahedral (αFEM-T4) elements with a scale factor α are formulated for two-dimensional (2D) and three-dimensional (3D) problems, respectively. The essential idea of the method is the use of a scale factor $\alpha \in [0,1]$ to obtain a combined model of the standard fully compatible model of the FEM and a quasi-equilibrium model of the node-based smoothed FEM (N-SFEM). This novel combination of the FEM and N-SFEM makes the best use of the upper bound property of the N-SFEM and the lower bound property of the standard FEM. Using meshes with the same aspect ratio, a unified approach has been proposed to obtain a nearly exact solution in strain energy for linear problems. The proposed elements are also applied to improve the accuracy of the solution of nonlinear problems of large deformation. Numerical results for 2D (using αFEM-T3) and 3D (using αFEM-T4) problems confirm that the present method gives the much more accurate solution comparing to both the standard FEM and the N-SFEM with the same number of degrees of freedom and similar computational efforts for both linear and nonlinear problems.

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1. Introduction

For many decades, the constant finite elements such as the three-node triangle and four-node tetrahedron are popular and widely used in practical. The reason is that these elements can be easily formulated and implemented very effectively in the finite element programs using piecewise linear approximation. Furthermore, most FEM (finite element method) codes for adaptive analyses are based on triangular and tetrahedral elements, due to the simple fact that triangular and tetrahedral meshes can be automatically generated.

However, these elements possess significant shortcomings, such as poor accuracy in stress solution, the overly stiff behavior and volumetric locking for plane strain problems in the nearly incompressible cases. In order to overcome these disadvantages, some new finite elements were proposed. For the triangular elements, Allman [1,2] introduced rotational degrees of freedom at the element nodes to achieve an improvement for the overly stiff behavior. Elements with rotational degrees of freedom were also considered in Ref. [3,4]. Piltner and Taylor [5] combined the rotational degrees of freedom and enhanced strain modes to give a triangular element which can achieve a higher convergence in energy and deal with the nearly incompressible plane strain problems. However, using more degrees of freedom at the nodes limits the practical application of those methods. For both triangular and tetrahedral elements, Dohrmann et al. [6] presented a weighted least-squares approach in which a linear displacement field is fit to an element's nodal displacements. The method is claimed to be computationally efficient and avoids the volumetric locking problems. However, more nodes are required on the element boundary to define the linear displacement field. Dohrmann et al. [7] also proposed a nodal integration finite element method (NI-FEM) in which each element is associated with a single node and the linear interpolation functions of the original mesh are used. The method avoids the volumetric locking problems and performs better comparing to standard triangular and tetrahedral elements in terms of stress solution for static problems.

In the other front of development, a conforming nodal integration technique has been proposed by Chen et al. [8] to stabilize the solutions in the context of the meshfree method and then applied in the natural-element method [9]. Liu et al. have applied this technique to formulate the linear conforming point interpolation method (LC-PIM) [10], the linearly conforming radial point

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interpolation method (LC-RPIM) [11]. Applying the same idea to the FEM, an element-based smoothed finite element method (SFEM) [12,13,43] and a node-based smoothed finite element method (N-SFEM) [14] have also been formulated. When only the linear shape function for interpolation is used, the LC-PIM is identical to the NI-FEM or N-SFEM using triangular and tetrahedral elements [14]. Liu et al. [15] have provided an intuitive explanation and showed numerically that when a reasonably fine mesh is used, the LC-PIM has an upper bound in the strain energy. The same finding is obtained for LC-RPIM and N-SFEM, meaning that the LC-RPIM and N-SFEM also have the similar upper bound property.

Obtaining exact solution measured in a norm using a numerical method is a fascinating idea in the area of computational methods. So far, the mixed FEM models [16–19] based on the mixed variational principles focus mainly to improve the accuracy of the solution. Recently, an alpha finite element method (αFEM) using four-node quadrilateral elements has been developed for the purpose of finding the nearly exact solution in strain energy even for the coarse mesh [20,21]. The αFEM is a novel FEM in which the gradient of strains is scaled by a factor $\alpha \in [0,1]$, and the coding of the αFEM is almost exactly the same as the standard FEM. The obtained result of strain energy is a continuous function of α between the solutions of the standard FEM using reduced integration and that using full Gauss integration. The significance of this formulating is two folds: (1) For overestimation problems, there exists an $\alpha \in [0,1]$ at which the solutions of αFEM is nearly exact in energy norm; (2) For underestimation problems, the αFEM solution obtained at $\alpha = 0$ is the closest to the exact solution in energy norm [20,21]. Based on the function of strain energy curves and the use of meshes with the same aspect ratio, a general procedure of the αFEM has been suggested to obtain the exact or best possible solution for a given problem: an exact-α approach is devised for overestimation problems; and a zero-α approach for underestimation problems. The αFEM has clearly opened a novel window of opportunity to obtain numerical solutions that are exact in certain norms. However, the αFEM based on quadrilateral elements cannot provide exact solution to all problems. Furthermore, the use of four-node quadrilateral elements in αFEM requires a quadrilateral mesh that cannot be generated in a full automated manner for complicated domains.

Making use of the upper bound property of the N-SFEM, the lower bound property of the standard FEM in the strain energy, and the importance idea of the αFEM for the four-node quadrilateral elements, we propose a novel alpha finite element method using three-node triangular (αFEM-T3) elements for 2D problems and four-node tetrahedral elements (αFEM-T4) for 3D problems. The essential idea of the method is to introduce a scale factor $\alpha \in [0,1]$ to establish a continuous function of strain energy that contains contributions from both the standard FEM and the N-SFEM. Our formulation ensures the variational consistence and the compatibility of the displacement field, and hence guarantees reproducing linear field exactly. Based on the fact that the standard FEM of triangular and tetrahedral elements is stable (no spurious zero energy modes), and so is the N-SFEM as proved by Liu et al. [14], our αFEM will be always stable. This stability ensures the convergence of the solution. Furthermore, this novel combined formulation of the FEM and N-SFEM makes the best use of the upper bound property of the N-SFEM and the lower bound property of the standard FEM. Using meshes with the same aspect ratio, a unified approach has been proposed to obtain the nearly exact solution in strain energy for a given linear problem. The proposed elements are also applied to nonlinear problems of large deformation. In such cases, the exact solution is usually difficult to obtain, but the accuracy of the solution can be significantly improved. Numerical results for 2D (using αFEM-T3) and 3D (using αFEM-T4) problems confirm that the present method gives the excellent performance comparing to both the standard FEM and the N-SFEM. It is very easy to implement and apply to practical problems of complicated geometry.

Note that the present αFEM-T3 and αFEM-T4 are very much different from the αFEM for quadrilateral elements (or αFEM-Q4) given in Ref. [20,21] in terms of both formulation procedures and the approach. First, the αFEM-Q4 is element based and αFEM-T3 (or αFEM-T4) is both element and node based; Second, in the case of αFEM-Q4, the strain field in the element is linear, which allows us to scale the gradient of the strain field by introducing a scaling factor α. In the present αFEM-T3 (or αFEM-T4), the strain field in the element is constant, and hence it is not possible to scale the gradient of the strain field. Therefore, a new technique has to be devised to create a desirable strain field; Third, αFEM-Q4 can only give nearly exact solution in strain energy for overestimation problems [20,21], while the present αFEM-T3 (or αFEM-T4) can provide nearly exact solution in strain energy for all linear problems without any post processing techniques.

The paper is outlined as follows. In Section 2, the idea the αFEM-T3 and αFEM-T4 is briefly introduced. In Section 3, some theoretical properties of the αFEM-T3 and αFEM-T4 are presented. Numerical implementations are described in Section 4 and patch testes are performed in Section 5. In Section 6, some numerical examples are examined and discussed to verify the formulations and properties of the αFEM-T3 and αFEM-T4. Some concluding remarks are made in the Section 7.

2. The idea of the present αFEM

2.1. *Briefing on the finite element method (FEM) [22–26]*

The discrete equations of the FEM are generated from the Galerkin weak form

$$\int_{\Omega}(\nabla_s\delta\mathbf{u})^{\mathrm{T}}\mathbf{D}(\nabla_s\mathbf{u})\,\mathrm{d}\Omega - \int_{\Omega}\delta\mathbf{u}^{\mathrm{T}}\mathbf{b}\,\mathrm{d}\Omega - \int_{\Gamma_t}\delta\mathbf{u}^{\mathrm{T}}\bar{\mathbf{t}}\,\mathrm{d}\Gamma = 0, \tag{1}$$

where $\mathbf{b}$ is the vector of external body forces, $\mathbf{D}$ is a symmetric positive definite (SPD) matrix of material constants, $\bar{\mathbf{t}}$ is the prescribed traction vector on the natural boundary Γ_t, $\mathbf{u}$ is trial functions, $\delta\mathbf{u}$ is test functions and $\nabla_s\mathbf{u}$ is the symmetric gradient of the displacement field.

The FEM uses the following trial and test functions

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I=1}^{\mathrm{NP}}\mathbf{N}_I(\mathbf{x})\mathbf{d}_I; \quad \delta\mathbf{u}^h(\mathbf{x}) = \sum_{I=1}^{\mathrm{NP}}\mathbf{N}_I(\mathbf{x})\delta\mathbf{d}_I, \tag{2}$$

where NP is the number of the nodal variables of the element, $\mathbf{d}_I$ is the nodal displacement vector, and $\mathbf{N}_I(\mathbf{x})$ is a matrix of shape functions.

By substituting the approximations, $\mathbf{u}^h$ and $\delta\mathbf{u}^h$, into the weak form and invoking the arbitrariness of virtual nodal displacements, Eq. (1) yields the standard discretized algebraic equation system:

$$\mathbf{K}^{\mathrm{FEM}}\mathbf{d} = \mathbf{f}, \tag{3}$$

where $\mathbf{K}^{\mathrm{FEM}}$ is the system stiffness matrix, $\mathbf{f}$ is the force vector, that are assembled with entries of

$$\mathbf{K}_{IJ}^{\mathrm{FEM}} = \int_{\Omega_e}\mathbf{B}_I^{\mathrm{T}}\mathbf{D}\mathbf{B}_J\,\mathrm{d}\Omega, \tag{4}$$

$$\mathbf{f}_I = \int_{\Omega_e}\mathbf{N}_I^{\mathrm{T}}(\mathbf{x})\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t}\mathbf{N}_I^{\mathrm{T}}(\mathbf{x})\bar{\mathbf{t}}\,\mathrm{d}\Gamma. \tag{5}$$

In Eq. (4), the *strain matrix* is defined as

$$\mathbf{B}_I(\mathbf{x}) = \nabla_s N_I(\mathbf{x}) \tag{6}$$

that produces *compatible* strain fields. Using the triangular and tetrahedral elements with the linear shape functions, the strain

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