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On the thermomechanical consistency of the time differential dual-phase-lag models of heat conduction

Stan Chiriţă [a,b,*], Michele Ciarletta [c], Vincenzo Tibullo [c]

[a] Faculty of Mathematics, Al. I. Cuza University of Iaşi, 700506 Iaşi, Romania
[b] Octav Mayer Mathematics Institute, Romanian Academy, 700505 Iaşi, Romania
[c] University of Salerno, via Giovanni Paolo II n. 132, 84084 Fisciano, SA, Italy

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ABSTRACT

This paper deals with the time differential dual-phase-lag heat transfer models aiming, at first, to identify the eventually restrictions that make them thermodynamically consistent. At a first glance it can be observed that the capability of a time differential dual-phase-lag model of heat conduction to describe real phenomena depends on the properties of the differential operators involved in the related constitutive equation. In fact, the constitutive equation is viewed as an ordinary differential equation in terms of the heat flux components (or in terms of the temperature gradient) and it results that, for approximation orders greater than or equal to five, the corresponding characteristic equation has at least a complex root having a positive real part. That leads to a heat flux component (or temperature gradient) that grows to infinity when the time tends to infinity and so there occur some instabilities. Instead, when the approximation orders are lower than or equal to four, this is not the case and there is the need to study the compatibility with the Second Law of Thermodynamics. To this aim the related constitutive equation is reformulated within the system of the fading memory theory, and thus the heat flux vector is written in terms of the history of the temperature gradient and on this basis the compatibility of the model with the thermodynamical principles is analyzed.

1. Introduction

The dual-phase-lag model of heat conduction proposed in [1–3] distinguishes the time instant $t+\tau_q$, at which the heat flux flows through a material volume and the time instant $t+\tau_T$, at which the temperature gradient establishes across the same material volume:

$$q_i(\mathbf{x}, t+\tau_q) = -k_{ij}(\mathbf{x})T_{,j}(\mathbf{x}, t+\tau_T), \qquad \text{with} \quad \tau_q,\, \tau_T \geqslant 0. \tag{1}$$

The above constitutive equation states, synthesizing its meaning, that the temperature gradient $T_{,j}$ at a certain time $t+\tau_T$ results in a heat flux vector q_i at a different time $t+\tau_q$. In the above constitutive Eq. (1), besides the explicit dependence upon the spatial variable, we point out that q_i are the components of the heat flux vector, T represents the temperature variation from the constant reference temperature $T_0>0$ and k_{ij} are the components of the conductivity tensor; moreover, t is the time variable while τ_q and τ_T are the phase lags (or delay times) of the heat flux and of the temperature gradient, respectively. In particular, τ_q is a relaxation time connected to the fast-transient effects of thermal inertia, while τ_T is caused by microstructural interactions, such as phonon scattering or phonon-electron interactions [4]. In addition to the thermal conductivity, the phase lags τ_T and τ_q are treated as two additional intrinsic thermal properties characterizing the energy-bearing capacity of the material.

Eq. (1) describing the lagging behavior in heat transport, when coupled with the energy equation

$$-q_{i,i}(\mathbf{x}, t) + \varrho(\mathbf{x})r(\mathbf{x}, t) = a(\mathbf{x})\frac{\partial T}{\partial t}(\mathbf{x}, t), \tag{2}$$

displays two coupled differential equations of a delayed type. Due to the general time shifts at different scales, τ_T and τ_q, no general solution has been known yet. The refined structure of the lagging response depicted by equations (1) and (2), however, has been illustrated by Tzou [3] by expanding Eq. (1) in terms of the Taylor's series with respect to time:

* Corresponding author.
E-mail addresses: schirita@uaic.ro (S. Chiriţă), mciarletta@unisa.it (M. Ciarletta), vtibullo@unisa.it (V. Tibullo).

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$$q_i(\mathbf{x},t)+\frac{\tau_q}{1!}\frac{\partial q_i}{\partial t}(\mathbf{x},t)+\frac{\tau_q^2}{2!}\frac{\partial^2 q_i}{\partial t^2}(\mathbf{x},t)+\cdots+\frac{\tau_q^n}{n!}\frac{\partial^n q_i}{\partial t^n}(\mathbf{x},t)$$
$$=-k_{ij}(\mathbf{x})\left[T_{,j}(\mathbf{x},t)+\frac{\tau_T}{1!}\frac{\partial T_{,j}}{\partial t}(\mathbf{x},t)+\frac{\tau_T^2}{2!}\frac{\partial^2 T_{,j}}{\partial t^2}(\mathbf{x},t)+\cdots+\frac{\tau_T^m}{m!}\frac{\partial^m T_{,j}}{\partial t^m}(\mathbf{x},t)\right]. \quad (3)$$

An interesting discussion concerning this expansion has been developed by Tzou [3] when $n-m=0$ or $n-m=1$, relating the progressive interchange between the diffusive and wave behaviors.

We emphasize that the related time differential models obtained considering the Taylor series expansions of both sides of the Eq. (1) and retaining terms up to suitable orders in τ_q and τ_T (namely, first or second orders in τ_q and τ_T) have been widely investigated with respect to their thermodynamic consistency as well as to interesting stability issues and wave propagation (see, for example, [5–9]). However, the general form of the time differential dual-phase-lag model as given by (3) wasn't treated up to now, except for the paper by Quintanilla and Racke [10], where the spatial behavior is studied for solutions of the equation obtained by eliminating the heat flux vector between the constitutive Eq. (3) and the energy Eq. (2), provided $n=m$ or $n=m+1$.

The main purpose of this paper is to study the thermodynamical and mechanical consistency of the constitutive Eq. (3). We infer that the feasibility study of this constitutive equation greatly depends on the structure of the differential operators involved in its mathematical expression. In fact, if we consider the constitutive Eq. (3) as an ordinary linear differential equation in terms of the unknown function $q_i(t)$ (or, equivalently, in terms of the unknown function $T_{,i}(t)$) then we can observe that for $n\geqslant 5$ (or $m\geqslant 5$) it admits at least a complex root having a positive real part. That implies that $q_i(t)$ (or $T_{,i}(t)$) can tends to infinity when the time tends to infinity and so we are led to instability situations. On this way we conclude that the time differential dual-phase-lag model based on a constitutive equation of type (3) with $n\geqslant 5$ or $m\geqslant 5$ cannot be considered able to describe real mechanical situations. Instead, when $n=0,1,2,3,4$ and $m=0,1,2,3,4$ this is not the case and we have to study the thermodynamic consistency of the corresponding model. To this aim we follow [6,11] and we reformulate the constitutive Eq. (3) in such a way that the heat flux vector q_i depends on the history of the temperature gradient. In this sense we rewrite the Eq. (3) in the framework of Gurtin and Pipkin [12] and Coleman and Gurtin [13] fading memory theory, and on this basis we analyze the compatibility of the model with the thermodynamical principles. Precisely, the thermodynamic consistency of the model in concern is established when $(m,n)\in\{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2),(3,2),(2,3),(3,3),(3,4),(4,3),(4,4)\}$, provided appropriate restrictions are placed on the delay times.

2. Thermomechanical consistency of the model

In this Section we consider the Eq. (3) as an ordinary linear nonhomogeneous differential (in time variable) equation in terms of the heat flux vector components and observe that its characteristic equation is

$$\frac{1}{n!}\tau_q^n\lambda^n+\frac{1}{(n-1)!}\tau_q^{n-1}\lambda^{n-1}+\cdots+\frac{1}{2!}\tau_q^2\lambda^2+\frac{1}{1!}\tau_q\lambda+1=0. \quad (4)$$

This equation is connected with the partial sums of the Maclaurin series for the exponential function e^z and with the incomplete gamma function and its roots have been intensively studied in literature (see e. g. Eneström [14–17]). On the basis of the Eneström-Kakeya theorem it follows that all the roots of the Eq. (4) lie outside of the disk of radius $\frac{1}{\tau_q}$. Moreover, the Eq. (4) has no real root if n is even, while when n is odd, it has only one real root. However, here we are interested if this equation has at least a complex root with a positive real part. To this aim we outline the results obtained by Gábor Szegö [18] and Jean Dieudonné [19] who showed that the roots of the scaled exponential sum function approach the portion of the Szegö curve: $|z\exp(1-z)|=1$ within the unit disk as $n\to\infty$. Moreover, with the aim to visualize this result for $n\geqslant 5$ we recommend the simulation for the software package Wolfram Mathematica 11 presented in the Appendix (see also the Fig. 1).

Table 1
The values of $x=\tau_q\lambda$ with λ solution of the characteristic Eq. (4) for $n=1,2,3,4$.

n	$x=\tau_q\lambda$
1	-1
2	$-1.0\pm 1.0i$
3	-1.5961 $-0.70196\pm 1.8073i$
4	$-0.27056\pm 2.5048i$ $-1.7294\pm 0.88897i$

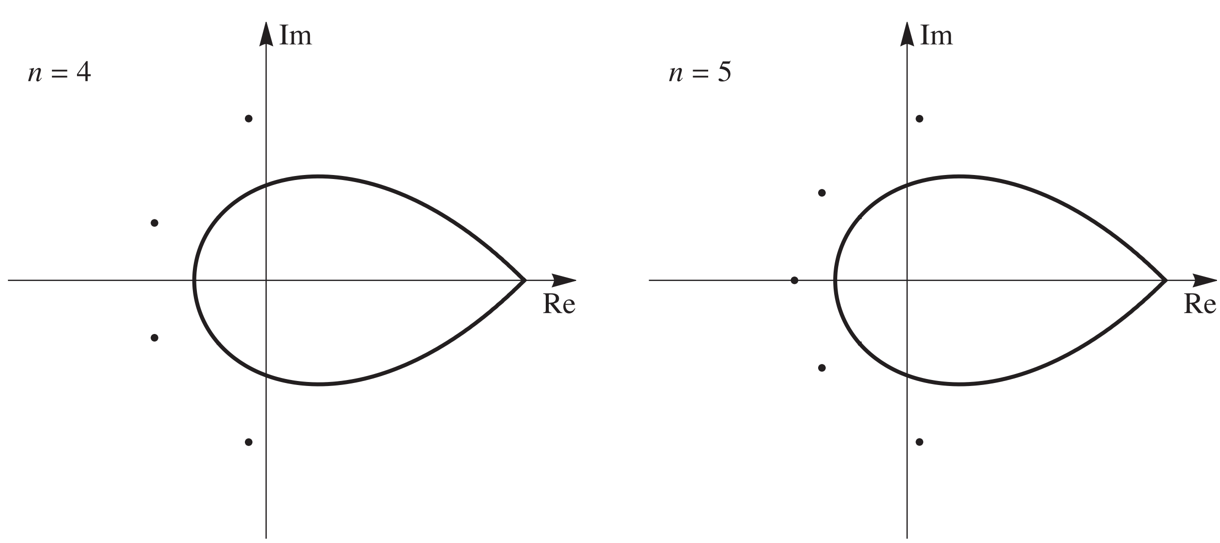

Fig. 1. Roots of the exponential sum for $n=4$ and for $n=5$.