International Journal of Heat and Mass Transfer 114 (2017) 407–418

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

An alternative theoretical approach for the derivation of analytic and numerical solutions to thermal Marangoni flows

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ARTICLE INFO

Article history:
Received 6 April 2017
Received in revised form 31 May 2017
Accepted 16 June 2017
Available online 26 June 2017

Keywords:
Marangoni flow
Analytic solutions
Two-dimensional and three-dimensional numerical results

ABSTRACT

The primary objective of this short work is the identification of alternate routes for the determination of exact and numerical solutions of the Navier-Stokes equations in the specific case of surface-tension driven thermal convection. We aim to elaborate a theoretical approach in which the typical kinematic boundary conditions required at the free surface by this kind of flows can be replaced by a homogeneous Neumann condition using a class of 'continuous' distribution functions by which no discontinuities or abrupt variations are introduced in the model. The rationale for such a line of inquiry can be found (1) in the potential to overcome the typical bottlenecks created by the need to account for a shear stress balance at the free surface in the context of analytic models for viscoelastic and other non-Newtonian fluids and/or (2) in the express intention to support existing numerical (commercial or open-source) tools where the possibility to impose non-homogeneous Neumann boundary conditions is not an option. Both analytic solutions and (two-dimensional and three-dimensional) numerical "experiments" (concerned with the application of the proposed strategy to thermocapillary and Marangoni-Bénard flows) are presented. The implications of the proposed approach in terms of the well-known existence and uniqueness problem for the Navier-Stokes equations are also discussed to a certain extent, indicating possible directions of future research and extension.

1. Introduction

Gravitational and surface-tension driven convection in fluid-filled enclosures has received considerable attention over the past several years due to its relevance to the engineering design of advanced technology. These applications (of practical or prototypical natures, too many to be cited here) span such diverse fields as electronic industry, cooling plants, coating processes, organic and inorganic crystal growth, etc. [1–5].

Leaving aside for a while applicative aspects, such subjects have also attracted significant *academic* interest. Indeed, the instabilities of these flows and related hierarchy of bifurcations are irresistible to researchers and scientists because of the variety of patterns and related spatio-temporal evolution. Such features are often aesthetically pleasing and "philosophically" challenging (because of their implications in the development of a general theory for the dynamics of non-linear systems). Much interest also comes from the well-known inherent difficulties in elaborating predictive models able to provide information on such characteristics "a priori".

One way to mitigate this drawback is to introduce a preliminary classification of the possible regimes and related solutions on the basis of the thermal and mechanical boundary conditions affecting the considered problem. A mathematical formulation of all such aspects leads to the so-called *initial-boundary value problem* (IBVP) where the governing balance equations for mass, momentum and energy have to be solved together with the related initial and boundary conditions. In turn, this requires implicitly the adoption of a given solution strategy, be it analytical, approximate or "numerical".

By analytical approach we mean one of the standard methods for classical partial differential equations (the obvious outcome of such a process being an algebraic expression relating the dependent variables to the independent variables, see, e.g., Ostroumov [6], Birikh [7,8], Gershuni and Zhukhovitskii [9], Belghazi et al. [10], Lappa [11], Lappa and Ferialdi [12]). An approximate method results when the governing boundary value problem is solved using a series-expansion-based technique (see, e.g., Jane [13], Jebari et al. [14], Al-Saif et al. [15]) or a transformation is used, based on the introduction of a similarity variable, by which the original partial differential equations are replaced by a set of coupled nonlinear ordinary differential equations (e.g., Makinde and Olanrewaju [16]). Finally, by numerical solution here we refer to

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http://dx.doi.org/10.1016/j.ijheatmasstransfer.2017.06.075
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the discrete set of nodal values that is obtained when the IBVP is integrated numerically (by discretizing the equations over a computational grid, see, e.g., [17–33]).

Unlike the analytical approach, however, none of these approximate numerical methods is able to yield a closed expression for the velocity in the flow in terms of driving forces involved and conditions at the system boundary.

This is the main reason for which analytic solutions of Marangoni convection have enjoyed a widespread use in the literature as a paradigm model for establishing (in general) a theoretical foundation to the field of surface-tension driven flows and (in particular) for explaining some of the typical manifestations of this kind of convection in practical situations.

These flows are known to undergo a variety of instabilities when the characteristic parameter (the so-called Marangoni number) exceeds one or more thresholds. Analytic solutions have allowed gaining outstanding insights into such behaviors due to their natural "ability" to be used as initial conditions (the so-called "basic flow") for straightforward application of linear-stability-analysis (LSA) techniques.

By contrast, when such a flow has to be determined numerically, the typical protocols of LSA require the solution of an eigenvalue problem of very high order (in practice, the order of this problem is equal to the amount of scalar unknowns used to represent the solution numerically, namely, a number given by the product of the number of unknown functions and the number of discretization elements, i.e. grid points, control volumes or finite elements effectively used).

Given such premises, it is really difficult to imagine how our understanding of these flows and their instabilities could have flourished without the availability of analytic solutions. Indeed, studies of buoyancy and Marangoni flow have historically progressed through the introduction of a precise hierarchy of models with increasing complexity where new knowledge was iteratively produced on the basis of the intuition developed on the basis of earlier simpler models. As an example, this approach proved extremely useful in the development of a general theory of flow bifurcation and turbulence [34–45].

Before being impressed by the power of this idea, however, we have to warn that the derivation of analytical solutions to the Navier-Stokes equations is an extremely hard task. Indeed, only a very limited set of exact solutions is known (most of such solutions were originally published in a number of Soviet-Union journals, hardly accessible in the western world, and for this reason many of them have been ignored for a long time, see, e.g., Ostroumov [6]).

Technically speaking, in general, it is possible to find or "build" analytical solutions when the convective terms in the governing equations, namely the main sources of non-linearity, vanish naturally. Towards this end, *some specific simplifications* can be considered such as a reduction of the number of space dimensions involved or the "removal" of physical boundaries along certain directions (ideally assumed to be located at an infinite distance where they are not able to influence the "core" flow).

Despite these assumptions, however, significant drawbacks opposing to the straightforward determination of results in analytical form persist. For Marangoni flow an additional bottleneck is represented by the need to satisfy a non-homogeneous kinematic boundary condition at the free interface, which should be regarded as the main motivation for which further developments in this field have been relatively limited. This is especially true for the case of non-Newtonian fluids, for which such a boundary condition (a shear stress balance) becomes particularly complex and cumbersome because of the presence of other (e.g., viscoelastic) stresses in the fluid (in addition to the standard Newtonian and thermocapillary ones). Among other things, the non-homogeneous nature of this boundary condition has acted in many circumstances as a kind of "barrier" limiting the utilization of widespread commercial or open-source CFD tools (such as OpenFoam). Though many of such computational tools are equipped with a variety of functions and models (including the possibility to simulate viscoelastic fluids), often they lack the possibility to implement kinematic boundary conditions such as those that would be required to simulate thermal Marangoni convection.

Motivated by this observational tide, the main aim of the present analysis is to propose widening the range of methodologies to be potentially used to treat this kind of flows. In particular, we further develop and expand the approach originally introduced by Tiwari and Nishino [46] about the possibility to turn the stress-balance Marangoni condition at the liquid-gas interface into an equivalent condition or source term to be added directly to the momentum equation as it was a force of buoyancy nature.

2. Governing equations

For simplicity, we build our framework on the assumption that the flow is laminar, steady and incompressible with constant properties.

2.1. Nondimensional form

In order to derive analytic solutions in the most general form, obviously, the governing equations have to be put in a non-dimensional shape. Here we consider the *typical* (most general) choice of characteristic reference quantities for thermal convection [47,48]; namely, we scale lengths, velocity, time and pressure by d, $V_\alpha = \alpha/d$, d^2/α and $\rho\alpha^2/d^2$, respectively, where d is a reference distance, α is the fluid thermal diffusivity (and ρ its density), and V_α is the energy diffusion velocity. Moreover, we subtract a reference value T_o to the temperature, while scaling it by a reference temperature difference ΔT. This approach leads to cast the mass, momentum and energy balance equations as:

$$\underline{\nabla} \cdot \underline{V} = 0 \tag{1}$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{\nabla} \cdot [\underline{V}\underline{V}] + \underline{\nabla} p = Pr \nabla^2 \underline{V} + \underline{F}_b \tag{2}$$

$$\frac{\partial T}{\partial t} + \underline{\nabla} \cdot [\underline{V} T] = \nabla^2 T \tag{3}$$

where $\underline{V}$, p and T are the non-dimensional velocity, pressure and temperature respectively (the so-called "primitive variables"), $\underline{F}_b$ is a generic body force (e.g., buoyancy), and Pr is the Prandtl number ($Pr = \nu/\alpha$ and ν is the constant fluid kinematic viscosity $\nu = \mu/\rho$).

To put the work in perspective, in the next two sections we illustrate the differences between classical approaches implemented in the past and the present one.

2.2. Classical analytic solutions for marangoni flows

As the resulting framework is not restricted to a specific geometry or model, without loss of generality we concentrate on the classical case of parallel flows [7,40].

As shown in Fig. 1, initially we consider a laterally unbounded horizontal layer of liquid delimited from below by a solid wall and from above by a liquid-gas interface. The horizontal boundaries are assumed to be located at $y = -1/2$ and $1/2$, respectively. Moreover, there are no velocity components along y and z ($v = w = 0$) while the component along x depends on the vertical coordinate y only, i.e. $u = u(y)$. The temperature undergoes a linear increase along the horizontal coordinate x (with a constant rate of