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A hybrid multilevel Schwarz method for the bidomain model

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ABSTRACT

A hybrid multilevel Schwarz method is studied numerically for the anisotropic Bidomain model in both two and three dimensions. This multiscale system models the electrical activity of the heart and it consists of two degenerate parabolic non-linear reaction–diffusion equations, coupled with a stiff system of ordinary differential equations. The numerical discretization of the whole system by finite elements in space and semi-implicit methods in time generates ill-conditioned linear systems that must be solved at each time step. The multilevel algorithm studied employs a hierarchy of nested meshes with overlapping Schwarz preconditioners on each level and is additive within the levels and multiplicative among the levels. We perform several parallel tests on two Linux clusters, showing that the convergence of the method is independent of the number of subdomains (scalability), the discretization parameters and the number of levels (optimality). Moreover the comparison with the traditional Block Jacobi ILU parallel preconditioner and the V-cycle Multigrid parallel preconditioner shows that, on a whole heart cycle simulation, the proposed method attains the best performances in terms of CPU times.

1. Introduction

A parallel multilevel solver for numerical simulations in computational electrocardiology is introduced and studied. The evolution of a complete heartbeat, from the excitation to the recovery phase, is modeled by the Bidomain system, a multiscale anisotropic model of the cardiac bioelectrical activity (see [29,39]). This model consists of a system of two degenerate parabolic partial differential equations, of reaction–diffusion type, describing at macroscopic level the intra and extracellular potentials of the myocardial tissue, coupled through the non-linear reaction term with a stiff system of ordinary differential equations, the so-called membrane model, which describes at microscopic level the ionic currents through the cellular membrane.

The numerical solution of this model is computationally very expensive, because of the ill-conditioning of the discrete system arising at each time step and the different space and time scales involved. It is worth noticing that the dimensions of meaningful portions of cardiac tissue is on the order of centimeters, while the accurate solution of the steep excitation front requires mesh sizes on the order of a tenth of millimeter. For what concerns the time scales, while a normal heartbeat is on the order of one second, the rapid kinetics in the membrane model require in some phases time steps on the order of the hundredths of milliseconds. Hence realistic models with three-dimensional uniform grids can yield simulations running for thousands of time steps and involving at each time step the solution of large-scale discrete problems.

Many different approaches (see e.g. [29]) have been developed in order to overcome these computational limits. Both parallel and adaptive solvers have been employed in literature. Fully implicit methods in time have been considered in few studies, see [19,18,17] and require the solution of non-linear systems at each time step; see also [12] for new developments on domain decomposition preconditioners for implicit methods. Most numerical studies employ semi-implicit methods in time that only require the solution of linear systems at each time step. Many different preconditioners have been proposed in order to devise efficient iterative solvers for such linear systems: diagonal preconditioners [27], Symmetric Successive Over Relaxation [33], Block Jacobi with incomplete LU factorization for each block [7,32], multigrid [3,30,34], multilevel additive Schwarz [26]. Parallel solvers based on finite difference discretizations can be found in [21,24]. Adaptive techniques are studied in [6].

The aim of this work is to study the numerical behavior of a hybrid multilevel Schwarz preconditioner, additive within the levels and multiplicative among the levels, applied to the Bidomain system. Parallel numerical results, both in two and three dimensions, using the PETSc library [4], show the scalability, the optimality and the efficiency of the multilevel method on large-scale simulations of a complete cardiac cycle. Moreover, in order to stress the performances of the preconditioner, we consider also ellipsoidal geometries and discontinuous diffusion coefficients, modeling pathological conditions such as myocardial ischemia. The comparison with the traditional Block Jacobi ILU parallel preconditioner

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and the V-cycle Multigrid parallel preconditioner shows that, to our knowledge, the multilevel hybrid Schwarz preconditioner for the Bidomain system attains the best performances to date in terms of CPU times, reducing the computational costs of about 78% and 12% compared to Block Jacobi and Multigrid, respectively.

The rest of the paper is organized as follows: in Section 2 we describe the mathematical models considered, in Section 3 we introduce the numerical methods used, and in particular the multilevel preconditioner, and finally in Section 4 we report the results of several parallel numerical tests.

2. Mathematical models

2.1. The anisotropic bidomain model

The cardiac ventricular tissue is modeled as an arrangement of fibers that rotate counterclockwise from epi- to endocardium, and that has a laminar organization modeled as a set of muscle sheets running radially from epi- to endocardium; see [15]. In the Bidomain model, the cardiac tissue is conceived as the superposition of two averaged continuous media, the intra and the extracellular medium (see e.g. [8,9]), whose anisotropy is characterized by the conductivity tensors $D_i(\mathbf{x})$ and $D_e(\mathbf{x})$, given by

$$D_{i,e} = \sigma_l^{i,e}\mathbf{a}_l\mathbf{a}_l^{\mathrm{T}} + \sigma_t^{i,e}\mathbf{a}_t\mathbf{a}_t^{\mathrm{T}} + \sigma_n^{i,e}\mathbf{a}_n\mathbf{a}_n^{\mathrm{T}}.$$

$\sigma_l^{i,e}, \sigma_t^{i,e}, \sigma_n^{i,e}$ are the conductivity coefficients measured along the corresponding directions $\mathbf{a}_l(\mathbf{x})$ (along fiber), $\mathbf{a}_t(\mathbf{x}), \mathbf{a}_n(\mathbf{x})$ (tangent and orthogonal to the radial laminae, respectively and both transversal to the fiber axis).

The intra and extracellular electric potentials u_i, u_e are described by a parabolic system of two reaction–diffusion partial differential equations (PDEs), coupled with a system of ODEs for ionic gating variables $w \in \mathbb{R}^M$ and for the ions concentration $c \in \mathbb{R}^S$. Denoting by $v = u_i - u_e$ the transmembrane potential then the Bidomain model for an insulated cardiac domain Ω can be written as the following reaction–diffusion system:

$$\begin{cases} c_{\mathrm{m}}\frac{\partial v}{\partial t} - \mathrm{div}(D_i\nabla u_i) + I_{\mathrm{ion}}(v,w,c) = 0 & \text{in } \Omega\times(0,T),\\ -c_{\mathrm{m}}\frac{\partial v}{\partial t} - \mathrm{div}(D_e\nabla u_e) - I_{\mathrm{ion}}(v,w,c) = -I_{\mathrm{app}}^e & \text{in } \Omega\times(0,T),\\ \frac{\partial w}{\partial t} - R(v,w) = 0, \quad \frac{\partial c}{\partial t} - S(v,w,c) = 0 & \text{in } \Omega\times(0,T),\\ \mathbf{n}^{\mathrm{T}}D_{i,e}\nabla u_{i,e} = 0 & \text{in } \partial\Omega\times(0,T),\\ v(\mathbf{x},0) = v_0(\mathbf{x}), \quad w(\mathbf{x},0) = w_0(\mathbf{x}), \quad c(\mathbf{x},0) = c_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1}$$

where $c_{\mathrm{m}} = \chi C_{\mathrm{m}}$, $I_{\mathrm{ion}} = \chi i_{\mathrm{ion}}$, with χ the ratio of membrane area per tissue volume, C_{m} the surface capacitance and i_{ion} the ionic current of the membrane per unit area. If the applied extracellular current I_{app}^e satisfies the compatibility condition $\int_\Omega I_{\mathrm{app}}^e\,\mathrm{d}x = 0$, this system uniquely determines v, while the potentials u_i and u_e are defined only up to a same additive time-dependent constant related to the reference potential, chosen to be the average extracellular potential in the cardiac volume by imposing $\int_\Omega u_e\mathrm{d}x = 0$. We refer to [8,9,20,31] for a mathematical analysis of the Bidomain model. In the following we will consider as membrane model the Luo–Rudy phase I model (LR1, [16]), briefly recalled in the next section.

2.2. The LR1 membrane model

Many models of Hodgkin-Huxley type have been developed for the cardiac cells, see e.g. [22]. In these models the ionic current through channels of the membrane, modulated by the transmembrane potential v, by gating variables $w := (w_1,\dots,w_M)$ and by ionic intracellular concentration variables $c := (c_1,\dots,c_S)$, is given by

$$I_{\mathrm{ion}}(v,w,c) = \sum_{k=1}^{N} G_k(v,c)\prod_{j=1}^{M} w_j^{p_{j_k}}(v - v_k(c)),$$

where N is the number of ionic currents, G_k is the membrane conductance and v_k the reversal potential for the kth current, p_{j_k} are integers. The dynamics of the gating variables w is described in the Hodgkin-Huxley formalism by a system of ODE's having the following structure

$$\begin{cases} \frac{\mathrm{d}w}{\mathrm{d}t} = R(v,w), \quad w(0) = w_0,\\ R_j(v,w_j) = \alpha_j(v)(1-w_j) - \beta_j(v)(w_j),\\ \alpha_j,\ \beta_j > 0,\ 0 \leqslant w_j \leqslant 1,\ j = 1,\dots,M. \end{cases} \tag{2}$$

The dynamics of the ionic concentration variables c is described by the additional system of ODE's

$$\begin{cases} \frac{\mathrm{d}c}{\mathrm{d}t} = S(v,w,c), \quad c(0) = c_0,\\ S_j(v,w,c_j) = -\frac{I_{c_j}(v,w)\cdot A_{cap}}{V_{c_j}\cdot z_{c_j}\cdot F},\ j = 1,\dots,S, \end{cases}$$

where I_{c_j} is the sum of ionic currents carrying ion c_j, A_{cap} is the capacitive membrane area, V_{c_j} is the volume of the compartment where c_j is updated, z_{c_j} is the valence of ion c_j and F is the Faraday constant.

The LR1 model, developed in 1991 by Luo and Rudy [16] for the left ventricle cells, consists of six ionic currents ($N = 6$), six gating variables ($M = 6$) and one ionic concentration variable ($S = 1$), the intracellular calcium. See Fig. 1 for the time evolution of v, $w_1,\dots,w_6$, and c_1 at a given point of the cardiac domain and the Appendix.

We remark that during a heartbeat, the time course of the transmembrane potential v at each point of the ventricular tissue, also called action potential, displays mainly three phases having different time scales. The first is related to the excitation phase, also called depolarization, where v undergoes an abrupt temporal change lasting about 2 ms, followed by a fast exponential decay toward a plateau value. The second is the plateau phase, lasting from 40–50 ms to about 400 ms, according to the ionic model used and the type of propagating front considered. In this phase, v varies very little and slowly in comparison with the previous depolarization phase and the cardiac tissue is refractory, i.e. any applied stimulus does not elicit another action potential. The last is the recovery phase, also called repolarization, where v returns to the rest value during a period lasting about 20–50 ms, after which the tissue becomes excitable again.

3. Numerical methods

In the three-dimensional case, our domain Ω representing the left ventricle is modeled by a family of truncated ellipsoids described by the parameter equations

$$\begin{cases} x = a(r)\cos\theta\cos\phi & \phi_{\min} \leqslant \phi \leqslant \phi_{\max},\\ y = b(r)\cos\theta\sin\phi & \theta_{\min} \leqslant \theta \leqslant \theta_{\max},\\ z = c(r)\sin\theta & 0 \leqslant r \leqslant 1, \end{cases}$$

where $a(r) = a_1 + r(a_2 - a_1)$, $b(r) = b_1 + r(b_2 - b_1)$, $c(r) = c_1 + r(c_2 - c_1)$ and $a_i, b_i, c_i, i = 1,2$ are given coefficients determining the main axes of the ellipsoid. We assume that the fibers rotate intramurally linearly with the depth for a total amount of 90° proceeding counterclockwise from epicardium to endocardium. Hence in a local ellipsoidal reference system $(\mathbf{e}_\phi, \mathbf{e}_\theta, \mathbf{e}_r)$ the fiber direction $\mathbf{a}_l(\mathbf{x})$ at a point $\mathbf{x}$ is given by

$$\mathbf{a}_l(\mathbf{x}) = \mathbf{e}_\phi\cos\alpha(r) + \mathbf{e}_\theta\sin\alpha(r), \quad \text{with } \alpha(r) = \frac{\pi}{2}(1-r) - \frac{\pi}{4},$$
$$0 \leqslant r \leqslant 1.$$

We also consider three-dimensional slabs of cardiac tissue, described in the usual canonic Cartesian coordinates system by