International Journal of Heat and Mass Transfer 115 (2017) 882–894

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Element differential method for solving general heat conduction problems

Xiao-Wei Gao *, Shi-Zhang Huang, Miao Cui, Bo Ruan, Qiang-Hua Zhu, Kai Yang, Jun Lv, Hai-Feng Peng

Dalian University of Technology, State Key Laboratory of Structural Analysis for Industrial Equipment, School of Aeronautics and Astronautics, Dalian 116024, China

ARTICLE INFO

Article history:
Received 9 June 2017
Received in revised form 23 July 2017
Accepted 13 August 2017
Available online 1 September 2017

Keywords:
Element differential method
EDM
Shape functions
Isoparametric elements
Heat conduction

ABSTRACT

In this paper, a new numerical method, Element Differential Method (EDM), is proposed for solving general heat conduction problems with variable conductivity and heat source subjected to various boundary conditions. The key aspect of this method is based on the direct differentiation of shape functions of isoparametric elements used to characterize the geometry and physical variables. A set of analytical expressions for computing the first and second order partial derivatives of the shape functions with respect to global coordinates are derived, which can be directly applied to governing differential equations and boundary conditions. A new collocation method is proposed to form the system of equations, in which the governing differential equation is collocated at nodes inside elements, and the flux equilibrium equation is collocated at interface nodes between elements and outer surface nodes of the problem. EDM is a strong-form numerical method. It doesn't require a variational principle or a control volume to set up the computational scheme, and no integration is involved. A number of numerical examples of two- and three-dimensional problems are given to demonstrate the correctness and efficiency of the proposed method.

1. Introduction

Heat conduction and other engineering problems are usually governed by a set of simultaneous second-order partial differential equations (PDEs), with a proper set of temperature (Dirichlet, essential) and flux (Neumann, natural) boundary conditions. Various numerical methods are available to solve these types of problems. The frequently used ones are the finite volume method (FVM) [1–4], finite difference method (FDM) [5–9], finite element method (FEM) [10–15], boundary element method (BEM) [16–22], and mesh free method (MFM) [23–28].

FVM and FDM are the frequently used methods in fluid mechanics as well as heat and mass transfer problems. FDM directly discretizes the governing differential equations based on the use of regularly distributed grids, while FVM works over the constructed control volume of each grid by employing the Gaussian integration principle to transform the flux-related terms in the governing equation to the boundary of the control volume. They have advantages of easily discretizing the governing equations and treating discontinuous physical phenomena, such as capturing shock waves and implementing upwind scheme. The main drawback of these two methods is that a lot of control volumes or points are required to achieve a satisfactory result, and the computational accuracy in heat flux on the boundary is poor [29–31].

FEM is the dominant method in the analysis of solid mechanics as well as other engineering problems. The distinct advantage of FEM is that almost any complicated engineering problems can be simulated using FEM. This feature is attributed to the use of various well-formed isoparametric elements which can be employed to discretize the geometry of the problem and interpolate physical variables. And, also because of using the isoparametric elements, the total numbers of elements and nodes required in FEM are much less than that used in FVM and FDM. The drawbacks of FEM are mainly embodied in the following aspects: (1) A variational or a virtual work principle is needed to establish the FEM formulation. Different problems have different representation forms of these principles, which gives rise to the inconvenience to set up a unified algorithm in handling multi-field coupling problems. (2) Domain integration for each of the elements is required, which sometimes is time consuming and different number of integration points (Gaussian points) may result in different computational accuracy of the variable gradients or stresses. (3) Too heavy distortion of the elements is prohibited to avoid the ill-condition of the element stiffness matrix.

* Corresponding author.
E-mail address: xwgao@dlut.edu.cn (X.-W. Gao).

http://dx.doi.org/10.1016/j.ijheatmasstransfer.2017.08.039
0017-9310/

It is worth mentioning that a new method, called the control volume finite element method (CVFEM), was proposed by Baliga and Patankar [32,33]. The CVFEM is a scheme that uses the advantages of both FVM for easy simulation of multi-physics problems and FEM for fitting complex geometries. This method has been well developed by Sheikholeslami and co-workers [34–37] for solving heat transfer and fluid dynamics.

Comparing to FEM, the BEM only needs to discretize the boundary of the problem into elements. This feature of BEM not only can reduce the dimensions of the problem by one, but also can easily simulate the heat and stress concentration behaviors [16,21]. The drawback of BEM is that a fundamental solution is required in setting up the BEM formulation, which is usually derived from a linear problem, and therefore it is difficult to establish a pure BEM algorithm for non-linear and non-homogeneous problems. Different from the above mentioned methods, MFM only needs a group of distributed points in the computational region, and therefore the distinct advantage of MFM is that problems with irregular geometries can be easily discretized. However, MFM has the drawbacks of time-consuming to form the global shape functions and difficulty to apply boundary conditions [23,24].

According to the formulation technique, the existing numerical methods can be roughly divided into two categories: weak-form and strong-form techniques [25]. In the weak-form technique, such as the FEM, BEM and part approaches of MFM, the governing PDE is solved indirectly, by converting it to a weak form using a mathematical principle, such as the variational principle, or an energy principle [10,12]. Attributing to the use of easily well-formed elements which can guarantee the variation of the physical variables to be consistent through element nodes, the weak-form technique usually results in very stable computational results.

In the strong-form technique, such as most approaches of MFM [25] and the conventional FDM, global shape functions are constructed by selecting a number of nodes around the node under consideration, and then, to form the final system of equations. The physical variables expressed in terms of the shape functions and their nodal values are directly substituted into the governing partial differential equations for each of the nodes in the interior of the domain, and into the relationships of boundary conditions about physical variables and their fluxes for all nodes on the boundary. Because all the equations (governing equations and boundary conditions) are all enforced at the nodes, this type of technique is usually called the collocation method [26,38]. The working process of this method is very straightforward, and can be easily coded for complicated multi-field problems. However, since there are no means to control the stability and the convergence of the solution, the collocation method is often found not stable and the solutions can vary a lot when the locations of the nodes change [25]. Besides, since the governing equations and the boundary conditions are entirely separately satisfied at individual nodes, there could be conflicts for nodes near the boundary. This disconnected situation could be one of the main causes of the instability issues in the collocation methods. Therefore, to ensure the stability of the solution, certain stabilization techniques must be used in the strong-form technique [23].

Recently, a different strong-form technique was proposed by Wen and Li [39–41], which is called the finite block method (FBM). A similar technique to FBM was proposed by Fantuzzi and Tornabene [42–44] for two-dimensional problems, which is called the strong formulation finite element method (SFEM). This type of methods incorporates the mapping technique proper of FEM and the strong form collocation approach. In these methods, isoparametric element-like blocks in FBM or sub-domains in SFEM are constructed based on the Lagrange interpolation formulation. In FBM, only the first order partial derivative of physical variables with respect to global coordinates is used and high order spatial derivatives are calculated through the recursive use of the first order derivative. In FBM and SFEM, the governing equation formulated using the constructed derivatives is applied to the internal nodes, and the specified boundary conditions on each block's/sub-domain's boundary are applied as independent equations. In this way, the formed final system of equations includes internal nodal temperatures and each block's boundary temperature and flux as unknowns. When solving a problem using FBM/SFEM, a few high order blocks, each having many nodes, are used to ensure the final system of equations not so large. As in other strong-form techniques, FBM/SFEM have the advantage of easy coding, but has the drawbacks of having too many unknowns in the system of equations and needing too many nodes over each block/sub-domain when solving a complicated practical engineering problem.

The motivation of this paper is to establish a numerical method which can be easily used as the collocation method and can result in stable computational results as FEM. To achieve this purpose, a new robust method, element differential method (EDM) [45], is proposed in this paper for solving general heat conduction problems based on the use of isoparametric elements as used in the standard FEM [4]. A set of explicit formulations of computing the first and second order spatial derivatives are derived for two-dimensional (2D) and three-dimensional (3D) problems. These formulations are expressed for shape functions of elements and therefore can be used to any physical variables' differentiation. EDM is a strong-form technique, which borrows the idea of FEM in the aspect of using isoparametric elements to obtain the spatial derivatives, and the idea of FBM/SFEM and collocation-like MFM in the aspect of collocating equations at nodes. The former (using elements) can result in very stable solutions and the latter (collocating at nodes) does not require any integration. Two distinct novelties can be found in the paper: (1) a set of analytical expressions of computing the second order spatial derivatives of shape functions for 3D problems are derived for the first time, which can make the computation more accurate and faster, and (2) a new collocation and assembling technique is proposed for forming the system of equations, which can make the system have the size as in the standard FEM, much smaller than in FBM and SFEM. Since EDM can use high order isoparametric elements to compute the spatial derivatives, the computational accuracy in heat flux is higher and the required total number of computational nodes are much less than the frequently used method (FVM) in heat transfer problems. The most important feature of the proposed method is that the derived spatial derivatives can be directly substituted into the governing equations and the heat flux equilibrium equations to form the final system of algebraic equations, and no any mathematical principles or integrations are required. Therefore, EDM is very easy to be coded in dealing with engineering problems with complicated governing equations and boundary conditions.

2. Governing equations for heat conduction problems

The governing equation for steady state heat conduction problems with a spatially varying thermal conductivity and heat source can be expressed as [27]

$$\nabla \cdot (\lambda \cdot \nabla T) + Q(x) = \frac{\partial}{\partial x_i}\left(\lambda_{ij}(x)\frac{\partial T(x)}{\partial x_j}\right) + Q(x) = 0 \tag{1}$$

The boundary conditions of the problem are

$$T(x) = \bar{T} \quad x \in \Gamma_1 \tag{2a}$$

$$q(x) = -\lambda_{ij}(x)\frac{\partial T(x)}{\partial x_j}n_i = \bar{q} \quad x \in \Gamma_2 \tag{2b}$$