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Stability of incompressible formulations enriched with X-FEM

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Abstract

The treatment of (near-)incompressibility is a major concern for applications involving rubber-like materials, or when important plastic flows occurs as in forming processes. The use of mixed finite element methods is known to prevent the locking of the finite element approximation in the incompressible limit. However, it also introduces a critical condition for the stability of the formulation, called the inf–sup or LBB condition. Recently, the finite element method has evolved with the introduction of the partition of unity method. The eXtended Finite Element Method (X-FEM) uses the partition of unity method to remove the need to mesh physical surfaces or to remesh them as they evolve. The enrichment of the displacement field makes it possible to treat surfaces of discontinuity inside finite elements. In this paper, some strategies are proposed for the enrichment of low order mixed finite element approximations in the incompressible setting. The case of holes, material interfaces and cracks are considered. Numerical examples show that for well chosen enrichment strategies, the finite element convergence rate is preserved and the inf–sup condition is passed.

Keywords: Mixed formulation; X-FEM; Partition of unity; Inf–sup condition; Incompressibility; Holes; Inclusions; Fracture mechanics

1. Introduction

Displacement-based finite element methods are nowadays abundantly used in engineering analysis. Indeed, they can solve a wide variety of problems, and have now been deeply mathematically investigated. However, there still exists two main drawbacks for these methods. First, the treatment of incompressible or nearly incompressible problems necessitates the use of adapted formulations. If not, incompressibility constraint locks the approximation, leading for instance to non-physical displacement or pressure fields. Second, the generation and especially the update of the mesh in complex 3D settings for evolving boundaries such as cracks, material interfaces and voids still lacks robustness, and involves important human effort.

Several techniques have been developed to respond to the locking issue. For instance, the selective-reduced-integration procedures [1–3] or the Bbar approach of Hughes [4] in which the volumetric part of the strain tensor is evaluated at the center of the element. Another way to avoid locking is to enhance the strain tensor in order to enlarge the space on which the minimization is performed, and meet the divergence-free condition (enhanced assumed strain methods, see [5–9]).

Here, we will focus on low order two-field mixed finite element methods. The incompressibility constraint is weakened by the introduction of the pressure field. This alleviates locking at the price of additional pressure unknowns. However, mixed finite element methods are not stable in all cases, some of them showing spurious pressure oscillations if displacement and pressure spaces are not chosen carefully. To be stable, a mixed formulation must verify consistency, ellipticity and the so called inf–sup (or LBB) condition. The latter is a severe condition which depends on the connection between the displacement and pressure approximation spaces. Stable mixed formulations can be obtained by stabilizing non-stable formulations with the use of parameters whose values may depend on the

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problem at hand. Otherwise, one has to work with approximation spaces which pass the inf–sup condition. To prove that a displacement–pressure pair satisfies the inf–sup condition is not a trivial task. However, a numerical test has been proposed by Brezzi and Fortin [10] and by Chapelle and Bathe [11], in order to draw a prediction on the fulfilment of the inf–sup. This test proved to be useful for the study of the stability of various mixed elements [12].

The second drawback of classical finite element methods (evolving boundaries) has been overcome by the development of alternative methods such as meshless methods in which the connectivity between the nodes is no longer obtained by the mesh, but by domains of influence which can be split by the boundaries. Moreover, the approximation basis can be enriched with functions coming from the a priori knowledge of the local character of the solution of the problem. Note that in the incompressible limit, meshless methods are now known to lock [13,14] as classical finite elements. Thus, some strategies have been developed to circumvent this issue [13,15]. Apart from mixing meshless methods and finite elements [16] another alternative to overcome the re-meshing issue in finite elements is to use partition of unity finite element methods which are based on the partition of unity concept introduced by Babuška and Melenk [17], and first employed in the context of the meshless method by Oden and Duarte [18,19]. Among the class of partition of unity finite element methods, the Generalized Finite Element Method (GFEM) and the eXtended Finite Element Method (X-FEM) are the most advanced. The GFEM was introduced by Strouboulis et al. [20–23] and was applied to the simulation of problems with complex micro-structures. The method was further extended to employ the idea of mesh-based numerically constructed handbook functions by Strouboulis et al. [24,25].

The X-FEM allows to model cracks, material inclusions and holes on non-conforming meshes, when used with proper enrichment of the finite element approximation. It has been applied to a wide variety of problems in fracture mechanics (2D [26–28], 3D [29–31], plates [32,33], cohesive zone modeling [34,35], dynamic fracture [36], non-linear fracture mechanics [37–39]), and in the study of heterogenous media (holes [40,41], material inclusions [41,42] and multiple phase flows [43]).

Here, we focus on the application of this method to mixed formulations for the treatment of holes, material inclusions and cracks in the incompressible limit. Bbar or selective-reduced formulations are not considered, because they do not seem to generalize easily to enriched displacement fields. The main contribution of this paper is the design of enrichment strategies for the pressure and displacement fields, so that it leads to a stable formulation. The enrichment of mixed finite element approximations has already been used by Dolbow et al. [33] and Areias et al. [39] for fracture mechanics in plates and shells, and by Wagner et al. [44] for rigid particles in Stokes flow. However, the stability and the convergence of these approaches was not studied. The latest work concerning volumetric incompressibility was proposed by Dolbow and Devan [37]. In this paper, the authors focus on the application of the enhanced assumed strain method to X-FEM in large strain. This approach seems to lead to a stable low order formulation in the case of nearly incompressible non-linear fracture mechanics. However, the stability of the method was not shown, and the influence of the near-tip enrichment was not studied. More precisely, it is not clear whether the near-tip enrichment could make this approach unstable, as the construction of an orthogonal enhanced strain field becomes difficult with non-polynomial functions.

The paper is organized as follows: first, the governing equations of incompressible linear elasticity are recalled. The conditions for the stability of mixed formulations are also reviewed. Next, some strategies are proposed to keep the stability of enriched finite elements. The case of holes, material interfaces and 2D cracks are presented. Finally, in a last section the stability of these strategies is investigated.

2. Governing equations

In this section, we focus on the design of stable mixed formulations for the treatment of incompressible elasticity under the assumption of small strain and displacement. First, the equations governing incompressible linear elasticity are recalled. Then the inf–sup condition is presented together with a numerical test.

2.1. Incompressible elasticity

We consider the static response of an elastic body (see Fig. 1) which occupies a bounded domain $\Omega \in \mathbb{R}^2$ with a sufficiently smooth boundary $\partial\Omega$ which is split into two disjoint parts: $\partial\Omega_u$ where displacements are prescribed (Dirichlet boundary conditions) and $\partial\Omega_T$ where tractions are prescribed (Neumann boundary conditions). The body is initially in an undeformed, unstressed state. The governing equations are

$$\begin{cases} \mathbf{div}\underline{\underline{\sigma}} + \mathbf{b} = \mathbf{0} \text{ on } \Omega, \\ \underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \text{ on} \Omega, \\ \underline{\underline{\sigma}} \cdot \mathbf{n} = \mathbf{T_d} \text{ on } \partial\Omega_T, \\ \mathbf{u}(\mathbf{x}) = \mathbf{u_d} \text{ on } \partial\Omega_u, \\ \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{\mathscr{C}}}}} : \underline{\underline{\varepsilon}} \text{ on } \Omega, \end{cases} \tag{1}$$

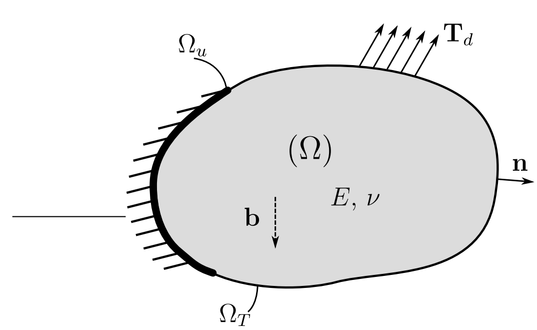

Fig. 1. The model problem.