International Journal of Heat and Mass Transfer 115 (2017) 1109–1120

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Modification of the fundamental theorem for transport phenomena in porous media

Yasuyuki Takatsu

Department of Intelligent Mechanical Engineering, Fukuoka Institute of Technology, 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka 811-0295, Japan

ARTICLE INFO

Article history:
Received 25 April 2017
Received in revised form 31 July 2017
Accepted 19 August 2017

Keywords:
Porous media
Volume averaging
Reynolds transport theorem
Governing equation

ABSTRACT

To examine convection in porous media, numerous analyses and numerical simulations have been conducted based on the macroscopic governing equations. In deriving the macroscopic governing equations, the phenomena intrinsic to porous media, such as Darcy's flow resistance, Forchheimer's flow resistance, and dispersion, are modeled using the theorem of the local volume average of a gradient (or a divergence). The theorem has been widely accepted as fundamental in the theory of convection in porous media; however, certain questions relating to the correctness of the theorem (the continuous derivative of the volume average and the pressure correction for Darcy's law) have been raised. In this study, we modify the conventional theorem for the local volume average of a gradient (or a divergence) to solve the aforementioned questions. First, we introduce the concept of a point mass to describe the reference point for the movement of the fluid phase, and we derive the Reynolds transport theorem in the macroscopic field that corresponds to the continuum of porous media. Then, we examine the definition of divergence at a point to obtain the relation between the microscopic description that employs the velocity vector $\mathbf{u}$ and quantity B of the fluid particle and the macroscopic description that employs the reference velocity vector $\mathbf{u_0}$ and the reference quantity B_0 of the point-mass particle, and we derive the modified theorem for the local volume average of a gradient (or a divergence). Furthermore, we derive the governing equations for porous media with the aid of the modified theorem.

1. Introduction

The local volume averaging method [1–3] is an important analytical technique for modeling the macroscopic governing equations that reflect the pore-level transports in porous media. This technique handles the macroscopic phenomena, such as Darcy's flow resistance [4], Forchheimer's flow resistance [5–6], dispersion [7–9], turbulence [10–13], and the boundary conditions at fluid–porous interfaces [14–18]. Slattery [19] proposed a theorem that replaces the average of a gradient with the gradient of an average. Before modifying this theorem, we will summarize its derivation [1–3]. The general transport theorem can be written as

$$\frac{d}{dt}\int_{V(t)} B dV = \int_{V(t)} \frac{\partial B}{\partial t} dV + \int_{A(t)} (B\mathbf{u_A}) \cdot \mathbf{n} dA, \tag{1}$$

where $V(t)$ is the volume bounded by surface $A(t)$, $\mathbf{u_A} = d\mathbf{r}/dt$ is the velocity of the area element dA, $\mathbf{n}$ is the normal unit vector to $A(t)$, $\mathbf{r}$ is the spatial position vector, and t is the time. Eq. (1) is applicable to any quantity B (tensor of any order), and the velocity of the area element, $\mathbf{u_A}$ can be different from the velocity of the material element (particles of fluids or solids), $\mathbf{u}$. When $\mathbf{u_A}$ and $\mathbf{u}$ are the same, then the general transport theorem becomes the Reynolds transport theorem which is used in formulating the basic (governing) equations of continuum mechanics

$$\frac{d}{dt}\int_{V(t)} B dV = \int_{V(t)} \frac{\partial B}{\partial t} dV + \int_{A(t)} (B\mathbf{u}) \cdot \mathbf{n} dA. \tag{2}$$

Here, d/dt becomes the material time derivative D/Dt and the total mass of material in the material volume $V(t)$ is conserved, independent of time [20,21]. The following expression, which was obtained by replacing time parameter t with arc length s in the general transport theorem, is applicable to any fluid-related quantity B in porous media:

$$\frac{d}{ds}\int_{V_f(s)} B dV = \int_{V_f(s)} \frac{\partial B}{\partial s} dV + \int_{A_f(s)} B\frac{d\mathbf{r}}{ds} \cdot \mathbf{n} dA, \tag{3}$$

where $V_f(s)$ is the volume of the fluid phase in a representative elementary volume $V_r = V_f + V_s$, $A_f(s)$ is the bounding surface of $V_f(s)$, and $V_s(s)$ is the volume of the rigid solid phase bounded by surface $A_s(s)$ (see Fig. 1). Howes and Whitaker [22] derived Eq. (3) directly rather than as an appendage of the general transport theorem [Eq. (1)], and the derivation is similar to that of the Reynolds transport

E-mail address: takatsu@fit.ac.jp

http://dx.doi.org/10.1016/j.ijheatmasstransfer.2017.08.067
0017-9310/

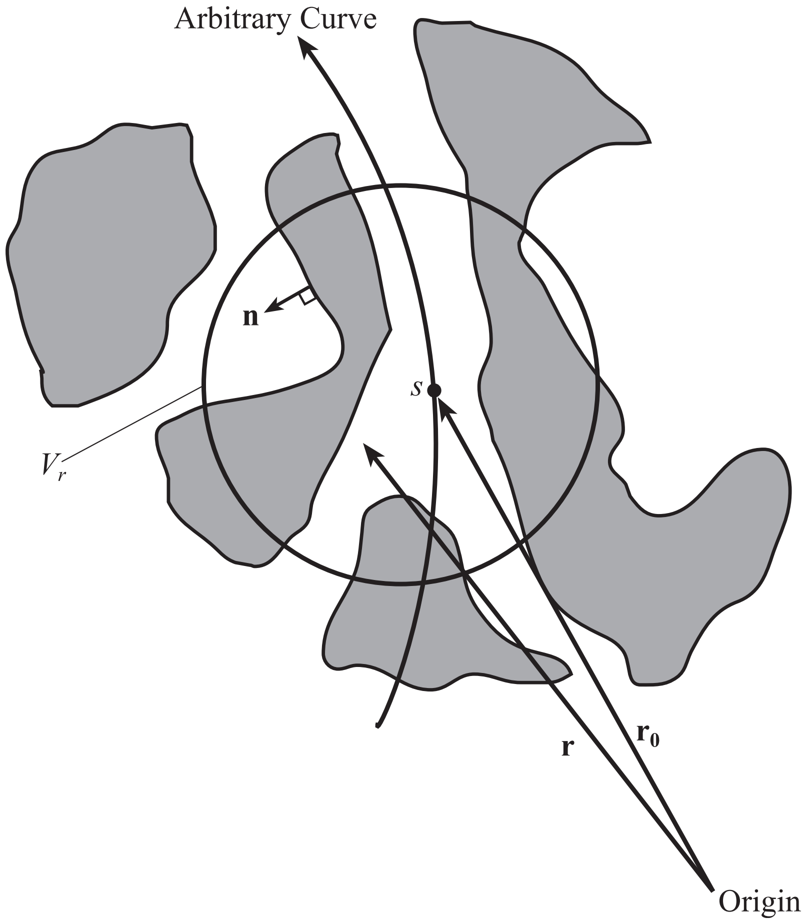

Fig. 1. Schematic of a representative elementary volume. The vector $\mathbf{r}_0$ is the position vector locating the reference point s on the arbitrary curve. Gray and white areas denote solid and fluid phases, respectively.

theorem with the use of the Jacobian of the transformation of variables [23]. The bounding surface A_f is divided into

$$A_f = A_{ff} + A_{fs}, \tag{4}$$

where A_{ff} is the area of the fluid phase passing through the surface of volume V_r, and A_{fs} the interfacial area between the fluid and the solid phases in V_r. To obtain the theorem for the local volume average of a gradient, the following three conditions are used:

(a) $B = B(x_i(s), t)$ is an explicit function of the spatial coordinates and time:

$$\frac{\partial B}{\partial s} = 0, \tag{5}$$

(b) $d\mathbf{r}/ds$ over the solid–fluid interface is a tangent vector to interface A_{fs}:

$$\frac{d\mathbf{r}}{ds} \cdot \mathbf{n} = 0 \quad \text{on } A_{fs}, \tag{6}$$

(c) provided that volume $V_r(s)$ is translated without rotation along the arbitrary curve s, any differential variation in $\mathbf{p}(= \mathbf{r} - \mathbf{r_0})$ is a tangent vector to surface A_{ff}:

$$\frac{d\mathbf{r}}{ds} \cdot \mathbf{n} = \frac{d\mathbf{r}_0}{ds} \cdot \mathbf{n} \quad \text{on } A_{ff}, \tag{7}$$

Applying Eqs. (5) and (6) to Eq. (3), we obtain

$$\frac{d}{ds} \int_{V_f} B dV = \int_{A_{ff}} B \frac{d\mathbf{r}}{ds} \cdot \mathbf{n} dA, \tag{8}$$

As the directional derivative [24] is given as

$$\frac{d}{ds} = \frac{d\mathbf{r}_0}{ds} \cdot \nabla, \tag{9}$$

substituting Eqs. (7) and (9) into Eq. (8) yields

$$\frac{d\mathbf{r}_0}{ds} \cdot \nabla \int_{V_f} B dV = \int_{A_{ff}} B \frac{d\mathbf{r}_0}{ds} \cdot \mathbf{n} dA. \tag{10}$$

If we remove $d\mathbf{r}_0/ds$ from the integral sign,

$$\int_{A_{ff}} B \frac{d\mathbf{r}_0}{ds} \cdot \mathbf{n} dA = \frac{d\mathbf{r}_0}{ds} \cdot \int_{A_{ff}} B\mathbf{n} dA, \tag{11}$$

Eq. (10) will be reduced to

$$\frac{d\mathbf{r}_0}{ds} \cdot \nabla \int_{V_f} B dV = \frac{d\mathbf{r}_0}{ds} \cdot \int_{A_{ff}} B\mathbf{n} dA. \tag{12}$$

As $d\mathbf{r}_0/ds$ is an arbitrary vector, we obtain the relation

$$\nabla \int_{V_f} B dV = \int_{A_{ff}} B\mathbf{n} dA, \tag{13}$$

By applying the divergence theorem [1,24], we obtain

$$\int_{V_f} \nabla B dV = \int_{A_{ff}} B\mathbf{n} dA + \int_{A_{fs}} B\mathbf{n} dA, \tag{14}$$

where the first term on the right-hand side of Eq. (14) is expressed as in Eq. (13); hence, the theorem for the local volume average of a gradient is given by

$$\langle \nabla B \rangle^{(f)} = \nabla \langle B \rangle^{(f)} + \frac{1}{V_f} \int_{A_{fs}} B\mathbf{n} dA, \tag{15}$$