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Stochastic modeling of random roughness in shock scattering problems: Theory and simulations

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ABSTRACT

Random roughness is omnipresent in engineering applications and may often affect performance in unexpected way. Here, we employ synergistically stochastic simulations and second-order stochastic perturbation analysis to study supersonic flow past a wedge with random rough surface. The roughness (of length d) starting at the wedge apex is modeled as stochastic process (with zero mean and correlation length A) obtained from a new stochastic differential equation. A multi-element probabilistic collocation method (ME-PCM) on *sparse grids* is employed to solve the stochastic Euler equations while a WENO scheme is used to discretize the equations in two spatial dimensions. The perturbation analysis is used to verify the stochastic simulations and to provide insight for small values of A, where stochastic simulations become prohibitively expensive. We show that the random roughness enhances the lift and drag forces on the wedge beyond the rough region, and this enhancement is proportional to $(d/A)^2$. The effects become more pronounced as the Mach number increases. These results can be used in designing smart rough skins for airfoils for maximum lift enhancement at a minimum drag penalty.

1. Introduction

Virtually any surface can be considered as rough under some fine-scale spatial resolution. Roughness poses naturally a multi-scale modeling problem as the characteristic large scale is often orders of magnitude greater than the roughness height ϵ or correlation length A. Attempting to model roughness in fluid mechanics applications often leads to either over-simplified formulations or prohibitively expensive simulations, so no systematic numerical studies have been published to date. In particular, for supersonic flow past aerodynamic objects with random rough surface even experimental studies are very limited. An intriguing experimental finding published in the Russian literature [1] suggests that roughness enhances lift in airfoils; this was later confirmed by other experimental studies in USA [2] but the highest speeds tested were below the supersonic regime.

Supersonic flow past a smooth wedge is a classical aerodynamics problem, which has been studied extensively [3–7]. The shock path and pressure distribution can be obtained by simple analytical formulas [8]. However, complex shock dynamics is observed when considering a random rough wedge surface. Lighthill [9] and Chu [10] used first-order perturbation analysis to study weak interactions, whereby the shock wave is only slightly perturbed from its base configuration. The first-order theory is adequate only for very small roughness height and does not provide a measure of the *mean* extra forces induced by roughness since for zero *mean* height the first-order theory predicts zero mean forces. Here, we employ second-order stochastic perturbation theory coupled with stochastic numerical simulations to study the effect of *large* and *fine* random roughness on shock dynamics. The use of the perturbation analysis results is twofold: First, to properly verify the simulation results for small roughness height. Second, to cover the parameter space in the limit of very small values of the correlation length A for which the numerical simulations become prohibitively expensive.

Specifically, to deal with the random roughness, a stochastic mapping technique [11] is employed to transform the original governing equations defined on a *random domain* into stochastic differential equations defined on a *deterministic domain*. This allows us to employ well-developed theoretical techniques and recent numerical methods for solving stochastic differential equations in deterministic domains. In particular, a high-order probabilistic collocation method (PCM, [12]) is used to solve the stochastic Euler equations. PCM combines the strengths of Monte Carlo methods and stochastic Galerkin methods. By taking advantage of the existing theory on multivariate polynomial interpolations (see [13,14]), fast convergence is achieved using PCM, when the solutions possess sufficient smoothness in the random space. Additionally, implementation of PCM is straightforward, as it only requires

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solutions of the corresponding deterministic problems at pre-selected sampling points. The choice of these sampling or collocation points is based on the sparse grid obtained from the Smolyak algorithm [15]. Sparse grids offer high-order accuracy with convergence rate not as strongly dependent on dimensionality. In the current paper, we extend the stochastic collocation method to a multi-element version (ME-PCM), which is computationally more attractive.

The paper is organized as follows: In the next section, we present the stochastic differential equation that models surface roughness. In Section 3, we give the analytical solutions of the perturbed forces for a full semi-infinite wedge derived from second-order stochastic perturbation analysis. In Section 4, we introduce the high-order stochastic collocation methods on *sparse grids* for the two-dimensional stochastic Euler equations and also discuss the stochastic mapping for random roughness. In Section 5, we present the analytical results from the second-order stochastic perturbation analysis and numerical simulation results. We conclude in Section 6 with a few remarks. We also include five Appendices (A–E) that provide more details on the analytical results.

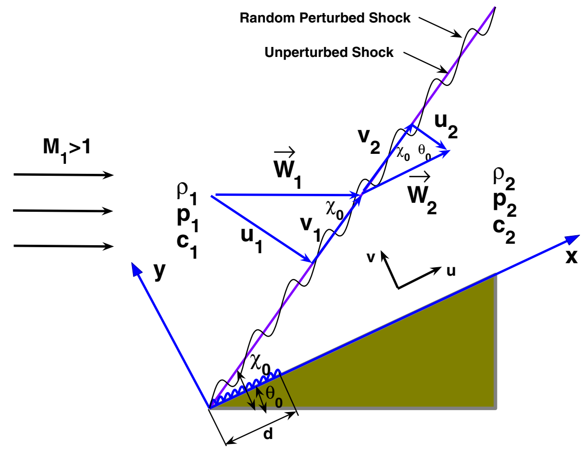

Fig. 1. Sketch of supersonic flow past a wedge with rough surface: Definition of coordinate system and notation; shown is also a perturbed shock path and the location of the unperturbed shock corresponding to a smooth wedge surface.

2. Modeling random roughness

We denote the roughness length as d, and we normalize all length except the correlation length A by the roughness length d. We describe the non-dimensional random roughness of correlation length A as a non-dimensional stochastic process $h_m(x;\omega)$ through the Karhunen–Loeve (KL) decomposition [16]:

$$h_m(x;\omega) = \overline{h}_m(x) + \sum_{i=0}^{\infty} \sqrt{\lambda_i}\psi_i(x)\xi_i(\omega), \tag{1}$$

where $\overline{h}_m(x)$ denotes the mean, $\{\xi_i(\omega)\}$ is a set of uncorrelated random variables with zero mean and unit variance, ω is a random event, and x is the spatial coordinate. Also, $\psi_i(x)$ and λ_i are the eigenfunctions and eigenvalues of the covariance kernel $R_{hh}(x_1,x_2)$, respectively, obtained from

$$\int_{\mathbf{D}} R_{hh}(x_1,x_2)\psi_i(x_2)\mathrm{d}x_2 = \lambda_i\psi_i(x_1). \tag{2}$$

We assume that $\overline{h}_m(x) = 0$, and the non-dimensional roughness height (distance from the smooth surface) is written as

$$y(x;\omega) = \epsilon h(x;\omega) = \epsilon\frac{h_m}{\mu}, \tag{3}$$

where $\mu = \max_x(\sigma(h_m))$, ϵ represents the amplitude of the non-dimensional roughness height, and $h = \frac{h_m}{\mu}$ is a second-order stochastic process with zero mean and unit variance. Here σ represents the standard deviation.

We can obtain the spatial covariance kernel $R_{hh}(h_m(x_1;\omega), h_m(x_2;\omega))$ (following the procedure in [17]) based on the solution of a fourth-order differential equation with stochastic right-hand-side, of the form:

$$\frac{\mathrm{d}^4 h_m}{\mathrm{d}x^4} + k^4 h_m = f(x), \tag{4}$$

where x is normalized by the roughness length d, $k = \frac{d}{A}$, and the random forcing term $f(x)$ is *white noise* satisfying: $\mathbb{E}[f(x_1)f(x_2)] = \delta(x_1 - x_2)$, where $\mathbb{E}[\cdot]$ denotes the expectation. Here we consider the case of a *finite strip* of roughness starting from the apex of the wedge and having length d, see Fig. 1. The required boundary conditions for this case are: $h_m(0;\omega) = h'_m(0;\omega) = h_m(1;\omega) = h'_m(1;\omega) = 0$. The corresponding covariance is given in Appendix A. The eigenfunctions and eigenvalues can be obtained as solutions of the homogenous equation $\frac{\mathrm{d}^4\psi}{\mathrm{d}x^4} - k^4\psi = 0$ with the boundary conditions $\psi(0) = \psi(1) = \psi'(0) = \psi'(1) = 0$. Such boundary conditions are chosen due to the assumption for second-order perturbation analysis, which assumes the random roughness and other perturbed quantities are small and smooth in the computational domain. The stochastic process $h_m(x;\omega)$ can then be represented by the KL expansion

$$h_m(x;\omega) = \sum_{n=1}^{\infty} \frac{1}{(\Lambda_n^4 + k^4)}\psi_n(x)\xi_n(\omega), \tag{5}$$

where $\psi_n(x) = \cos\Lambda_n x - \cosh\Lambda_n x - \frac{\cos\Lambda_n - \cosh\Lambda_n}{\sin\Lambda_n - \sinh\Lambda_n}(\sin\Lambda_n x - \sinh\Lambda_n x)$, Λ_n is obtained by solving $\cos\Lambda_n\cosh\Lambda_n = 1$, and $\{\xi_n(\omega)\}$ is a set of uncorrelated random variables with zero mean and unit variance. The stochastic perturbation analysis we develop can deal with random variables with different probability density functions. In the numerical results, we use primarily *uniform* random variables $\xi_n \in [-\sqrt{3},\sqrt{3}]$ and random variables with beta distributions: for $\alpha = 1$ and $\beta = 1$, $\xi_n \in [-\sqrt{5},\sqrt{5}]$; for $\alpha = 2$ and $\beta = 2$, $\xi_n \in [-\sqrt{7},\sqrt{7}]$; for $\alpha = 5$ and $\beta = 5$, $\xi_n \in [-\sqrt{13},\sqrt{13}]$.

In order to investigate the effect of roughness granularity, we study three different non-dimensional correlation lengths $A/d = 1$, $A/d = 0.1$ and $A/d = 0.01$. These values determine the number of random dimensions that are required for accurate representation of the random roughness through the KL expansion. Here we are using the following criterion:

$$\sum_{n=0}^{N} \frac{\Lambda_n}{(\Lambda_n^4 + k^4)} \geqslant 90\% \sum_{n=0}^{\infty} \frac{\Lambda_n}{(\Lambda_n^4 + k^4)}$$

based on which we choose the number of dimensions N. We arrived at this criterion after considerable testing. If the number of random dimensions is not sufficient, oscillations are observed for both the mean and the variance. For the results we present in this paper we have found that $N = 2$ is required for $A/d = 1$, $N = 12$ for $A/d = 0.1$ and $N = 60$ for $A/d = 0.01$.

3. Stochastic perturbation analysis

We consider the perturbation from the mean location of an oblique shock in supersonic flow past a rough half-wedge with a finite roughness strip of length d while the rest of the wedge is smooth; a schematic of this problem and notation are shown in Fig. 1. We assume that: (1) The random wedge roughness is small, and correspondingly the perturbation of the shock slope is small. (2) The oblique shock is attached to the wedge. (3) The flow between the shock and the wedge is adiabatic.

The domain of solution is between the perturbed shock and the wedge surface, and we employ the Rankine–Hugoniot conditions