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Multiscale finite element methods for stochastic porous media flow equations and application to uncertainty quantification

P. Dostert [a], Y. Efendiev [a,*], T.Y. Hou [b]

[a] Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, United States
[b] Applied Mathematics, Caltech, Pasadena, CA 91125, United States

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ABSTRACT

In this paper, we study multiscale finite element methods for stochastic porous media flow equations as well as applications to uncertainty quantification. We assume that the permeability field (the diffusion coefficient) is stochastic and can be described in a finite dimensional stochastic space. This is common in applications where the coefficients are expanded using chaos approximations. The proposed multiscale method constructs multiscale basis functions corresponding to sparse realizations, and these basis functions are used to approximate the solution on the coarse-grid for any realization. Furthermore, we apply our coarse-scale model to uncertainty quantification problem where the goal is to sample the porous media properties given an integrated response such as production data. Our algorithm employs pre-computed posterior response surface obtained via the proposed coarse-scale model. Using fast analytical computations of the gradients of this posterior, we propose approximate Langevin samples. These samples are further screened through the coarse-scale simulation and, finally, used as a proposal in Metropolis–Hasting Markov chain Monte Carlo method. Numerical results are presented which demonstrate the efficiency of the proposed approach.

1. Introduction

Many porous media processes are affected by heterogeneities at various length scales as well as uncertainties. To predict the flow and transport in stochastic porous media, some type of coarsening is needed. The upscaling and multiscale methods for a realization of heterogeneous porous media are extensively studied. In this paper, we present an approach for sampling permeability conditioned to an integrated response. Our proposed approach combines multiscale finite element methods with sparse collocation techniques representing uncertainty space. The goal of multiscale methods is to coarsen the flow equations spatially. A sparse collocation method is used to overcome the interpolation in high dimensional uncertainty space.

As for multiscale techniques, we use multiscale finite element type methods. A multiscale finite element method was first introduced in [22]. The main idea of multiscale finite element methods is to incorporate the small-scale information into finite element basis functions and couple them through a global formulation of the problem. The multiscale method in [22] shares some similarities with a number of multiscale numerical methods, such as residual free bubbles [8], variational multiscale method [23,3], two-scale conservative subgrid approaches [3], and multiscale mortar methods [4]. We remark that special basis functions in finite element methods have been used earlier in [6,5]. The multiscale finite element methodology has been modified and successfully applied to two-phase flow simulations in [24,1,14,17] and extended to nonlinear partial differential equations [19,17].

In this paper, we consider permeability fields generated using a two-point variogram. This type of permeability fields can be characterized using the Karhunen–Loève expansion which results in a parameterization of the uncertainty. Due to the high dimensional nature of uncertainty space, one can not resolve all realizations. We resort to sparse interpolation techniques (e.g., [31]) to represent the uncertainties and multiple scales. We note that approaches which combine sparse collocation techniques and multiscale methods are not new. In a recent paper [20], the authors propose an approach which uses variational multiscale method as well as multiscale finite element methods to solve a stochastic parabolic equation. In particular, the stochastic parabolic equation is solved using deterministic multiscale approaches at sparse collocation points in uncertainty space and then the solution is interpolated. To our best knowledge, this is the first approach which combines multiscale spatial methods and sparse collocation techniques. Our proposed approaches follow the main idea of this approach.[1] We discuss two type of approaches. In the first approach,

* Corresponding author.
E-mail address: efendiev@math.tamu.edu (Y. Efendiev).

[1] We were not aware of the paper [20] when we were writing our paper. We thank the reviewer for pointing it out.

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the basis functions are interpolated using pre-computed basis functions at collocation points. In the second approach, the stochastic solution is projected to a finite dimensional space consisting of all basis functions. Basis functions can be constructed both locally and globally. The latter is efficient when there is no scale separation. The difference between the proposed approach (the first approach) and the approach presented in [20] is that the proposed approach interpolates the basis functions which can be repeatedly used for different boundary conditions and source terms. The latter is important for porous media applications.

We apply the proposed technique to an uncertainty quantification problem where the permeability field is sampled based on oil production rates (an integrated response). This sampling is performed using Markov chain Monte Carlo (MCMC) methods with Langevin instrumental probability distribution. We first compute the posterior distribution at sparse locations that correspond to some selected realizations of the permeability field. These computations are performed on the coarse (spatial) grid, and thus they are inexpensive. Furthermore, the posterior distribution is approximated using polynomial interpolation. Based on interpolated posterior distribution, Langevin samples are proposed using analytical gradients of the posterior distribution. These samples are further screened with coarse-scale models. If the screening is passed, we perform fine-scale simulations to make a final acceptance decision.

The difference between the proposed approach and the previous findings (e.g., [12]) is that we use an approximate posterior distribution based on both interpolation in uncertainty space and coarsening in physical space. In [12], for each proposal, the corresponding fine-scale equations are coarsened and new permeability is proposed based on the coarse-scale models. This implies that one still needs to perform multiple coarse-scale simulations for each proposal in order to compute the gradient of the posterior distribution. Though the upscaling is in general inexpensive, performing many upscaled models will slow down the simulations. The main idea of the proposed method is to perform a few coarse-scale simulations *apriori* using multiscale methods and then use these results to interpolate the posterior distribution. Thus, each proposal is computed analytically once the posterior distribution is approximated. This procedure is faster than our previously proposed methods which is also shown numerically.

In the paper, we present numerical results to show the efficiency of the proposed approaches. We consider permeability fields prescribed by covariance matrix. Next, we use Karhunen–Loève expansion to parameterize the permeability field. Both normal and exponential variograms are considered. In the case of the latter, the uncertainty space is large (in order of 100 dimensions). This makes the uncertainty quantification CPU demanding since one needs to run many simulations. Numerical results show that the proposed algorithm is efficient and it has mixing properties similar to fine-scale Langevin algorithm. Moreover, we show that the proposed methods provide an order of magnitude CPU saving compared to the approaches where no interpolation is used.

This paper is organized as follows: In the next section, we briefly describe the model equations and known multiscale techniques for solving flow equations. In Section 3, we describe multiscale methods for the stochastic flow equations. In Section 4, the applications to uncertainty quantification is presented. The last two sections are dedicated to numerical implementation and numerical results.

2. Preliminaries

In this section, we present a model problem and background material for the multiscale finite element method. We consider two-phase flows in a reservoir (denoted by Ω) under the assumption that the displacement is dominated by viscous effects; i.e., we neglect the effects of gravity, compressibility, and capillary pressure. Porosity will be considered to be constant. The two phases will be referred to as water and oil, designated by subscripts w and o, respectively. We write Darcy's law for each phase as follows:

$$v_j = -\frac{k_{rj}(S)}{\mu_j}\boldsymbol{k}\cdot\nabla p, \tag{2.1}$$

where v_j is the phase velocity, $\boldsymbol{k}$ is the permeability tensor, k_{rj} is the relative permeability to phase j ($j = o, w$), S is the water saturation (volume fraction) and p is pressure. Throughout the paper, we will assume that the permeability tensor is diagonal $\boldsymbol{k} = k\boldsymbol{I}$, where k is a scalar and $\boldsymbol{I}$ is the unit tensor. In this work, a single set of relative permeability curves is used. Combining Darcy's law with a statement of conservation of mass allows us to express the governing equations in terms of the so-called pressure and saturation equations:

$$\nabla\cdot(\lambda(S)k\nabla p) = h, \tag{2.2}$$

$$\frac{\partial S}{\partial t} + v\cdot\nabla f(S) = 0, \tag{2.3}$$

where λ is the total mobility, h is the source term, $f(S)$ is the flux function, and v is the total velocity, which are respectively given by

$$\lambda(S) = \frac{k_{rw}(S)}{\mu_w} + \frac{k_{ro}(S)}{\mu_o}, \tag{2.4}$$

$$f(S) = \frac{k_{rw}(S)/\mu_w}{k_{rw}(S)/\mu_w + k_{ro}(S)/\mu_o}, \tag{2.5}$$

$$v = v_w + v_o = -\lambda(S)k\cdot\nabla p. \tag{2.6}$$

The above descriptions are referred to as the fine model of the two-phase flow problem. For the single-phase flow, $k_{rw}(S) = S$ and $k_{ro}(S) = 1 - S$.

For later discussion, we need to define the fractional flow response. Fractional flow (also referred as oil cut) is an integrated response over the whole domain. The oil cut (or fractional flow) is defined as the fraction of oil in the produced fluid and is given by q_o/q_t, where $q_t = q_o + q_w$, with q_o and q_w being the flow rates of oil and water at the production edge of the model. In particular, $q_w = \int_{\partial\Omega^{\text{out}}} f(S)v\cdot n\,d\omega$, $q_t = \int_{\partial\Omega^{\text{out}}} v\cdot n\,d\omega$, and $q_o = q_t - q_w$, where $\partial\Omega^{\text{out}}$ is the outer flow boundary. Pore volume injected, defined as $PVI = \frac{1}{V_p}\int_0^t q_t(\tau)\,d\tau$, with V_p being the total pore volume of the system, provides the dimensionless time for the displacement.

Next, we briefly describe the use of multiscale finite element methods for two-phase flow equations. We will use the multiscale finite element framework, though a finite volume element method is chosen as a global solver. Finite volume method is chosen because, by its construction, it satisfies the numerical local conservation which is important in groundwater and reservoir simulations. Let $\mathscr{K}^h$ denote the collection of coarse elements/rectangles K. Consider a coarse element K, and let ξ_K be its center. The element K is divided into four rectangles of equal area by connecting ξ_K to the midpoints of the element's edges. We denote these quadrilaterals by K_ξ, where $\xi \in Z_h(K)$, are the vertices of K. Also, we denote $Z_h = \bigcup_K Z_h(K)$ and $Z_h^0 \subset Z_h$ the vertices which do not lie on the Dirichlet boundary of Ω. The control volume V_ξ is defined as the union of the quadrilaterals K_ξ sharing the vertex ξ.

The key idea of the method is the construction of basis functions on the coarse-grids, such that these basis functions capture the small-scale information on each of these coarse-grids. The method that we use follows its finite element counterpart presented in [22]. The basis functions are constructed from the solution of the leading order homogeneous elliptic equation on each coarse element with some specified boundary conditions. We consider a coarse element K that has d vertices, the local basis functions $\phi_i, i = 1,\ldots,d$ are set to satisfy the following elliptic problem: