International Journal of Heat and Mass Transfer 108 (2017) 721–729

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

# Application of the meshless generalized finite difference method to inverse heat source problems

Yan Gu [a,*], Lei Wang [a,*], Wen Chen [b], Chuanzeng Zhang [c], Xiaoqiao He [d]

[a] School of Mathematics and Statistics, Qingdao University, Qingdao 266071, PR China
[b] Department of Engineering Mechanics, College of Mechanics and Materials, Hohai University, Nanjing 210098, PR China
[c] Department of Civil Engineering, University of Siegen, Paul-Bonatz-Str. 9-11, D-57076 Siegen, Germany
[d] Department of Civil and Architectural Engineering, City University of Hong Kong, Hong Kong

ARTICLE INFO

*Article history:*
Received 18 October 2016
Received in revised form 10 December 2016
Accepted 19 December 2016

*Keywords:*
Generalized finite difference method
Meshless method
Inverse heat source problem
Steady-state heat conduction

ABSTRACT

The generalized finite difference method (GFDM) is a relatively new domain-type meshless method for the numerical solution of certain boundary value problems. This paper documents the first attempt to apply the method for recovering the heat source in steady-state heat conduction problems. In order to guarantee the uniqueness of the solution, the heat source here is assumed to satisfy a second-order partial differential equation, and thereby transforming the problem into a fourth-order partial differential equation, which can be solved conveniently and stably by using the GFDM. Numerical analysis are presented on three benchmark test problems with both smooth and piecewise smooth geometries. The stability and sensitivity of the scheme with respect to the amount of noise added into the input data are analyzed. The numerical results obtained show that the proposed algorithm is accurate, computationally efficient and numerically stable for the numerical solution of inverse heat source problems.


## 1. Introduction

The finite element (FEM) and finite difference (FDM) methods have long been dominant numerical techniques in the simulation of real-world engineering applications. These methods, however, require the task of meshing the whole domain, which can be arduous, time-consuming and computationally expensive for certain classes of problems [1–3]. During the past two decades, some considerable effort was devoted to proposing novel computational algorithms that circumvent or greatly eliminate the problems associated with boundary or domain meshing. This led to the development of meshless or meshfree methods, which require neither domain nor boundary meshing. Generally, these methods can be divided into the domain-type [4,5] or boundary-type [6–12] techniques, depending on whether their basis functions satisfy the governing equation of interest. For an overview of the state of the art, we refer the reader to Refs. [7,13–17], as well as the references therein.

The generalized finite difference method (GFDM) belongs to the family of domain-type meshless methods and now has been successfully tried for many kind of engineering problems. In the GFDM, by utilizing the Taylor series expansions and weighted least-squares method, the derivatives of unknown variables can be expressed as linear combinations of function values of neighboring nodes. In addition, the concept of the *star* used in the GFDM yield a sparse matrix system, which makes the method very easy to implement by using standard sparse-matrix solvers. The basis of the method was published in the early eighties by Liszka and Orkisz [18,19] and were latter essentially improved and extended by many other authors. Now the most advanced version was given by Benito et al. in 2001 [20], including the point generation, local approximation and automatic selection of stars. In 2003, Gavete et al. [21] analyzed the influences of several factors on the accuracy of the GFDM, such as the shapes of the star, the number of nodes used in the star and different choices of the weighting functions. Their researches can be viewed as a good guidance for using the GFDM. In 2003, Benito et al. [22] proposed an $h$-adaptive algorithm to avoids incorrect stars and improve the accuracy of the method. In a more recent study, Ureña et al. [23] extended the GFDM to solve the third- and fourth-order partial differential equations. In recent years a few different meshless techniques have been proposed and developed which are different but also highly related to the GFDM method, such as the diffuse approximation method (DAM) proposed by Sadat et al. Interested readers are referred to Refs. [24–26] for details of the original and new versions of the DAM algorithms.

* Corresponding authors.
*E-mail addresses:* guyan1913@163.com (Y. Gu), shywanglei@126.com (L. Wang).

http://dx.doi.org/10.1016/j.ijheatmasstransfer.2016.12.084
0017-9310/

Inverse source problems arise in many branches of science and engineering [27,28], such as heat conduction, crack identification, electromagnetic theory, geophysical prospecting and pollutant detection. In inverse heat source problems, the interior heat source describing the direct problem is missing, due to technical difficulties associated with data acquisition. To fully determine the process, additional data must be supplied, either other boundary conditions on the same accessible part of boundary or measurements at some internal points in the domain. A formal mathematical model of an inverse problem can be derived with relative ease. However, the process of solving such problems is extremely difficult due to the fact that they are ill-posed in the sense that small errors in measured data may lead arbitrarily large changes in the numerical solution. The recovery of a general heat source presents difficulty and only a limited number of papers devoted to this subject are available in the literature [29].

In the present paper, we investigate a numerical scheme based on the GFDM for solving the inverse heat source problem associated with the steady-state heat conduction. It should be noted that the GFDM has recently been applied to inverse problems with great success, such as inverse biharmonic boundary value problems [30] and Cauchy problem for various partial differential equations [31]. To our knowledge, this is the first time the GFDM is applied for solving inverse heat source problems. It is worthwhile to mention that the interior heat source, in generally, cannot be determined uniquely by the boundary measurements [29]. The inverse source problem becomes solvable if some *a priori* knowledge is assumed. For instance, if the base area of a cylindrical source is known, the sought source is a characteristic function or a point source, then the unknown interior source can be uniquely determined by the boundary data. In the present study, the source is assumed to satisfy a second-order partial differential equation, and thereby transforming the problem into a fourth-order partial differential equation which can be conveniently solved using the GFDM.

A brief outline of the rest of this paper is as follows. Section 2 introduces the mathematical formulation of inverse heat source problems. The GFDM formulation and its numerical implementation for general fourth-order partial differential equations are presented in Section 3. Next, numerical analysis are presented in Section 4 on three benchmark test problems with both smooth and piecewise smooth geometries. Finally, some conclusions and remarks are provided in Section 5.

## 2. Mathematical formulation for inverse heat source problems

Let $\Omega$ be an open bounded domain and assume that $\Omega$ is bounded by a surface $\Gamma$ which may consist of several segments, each being sufficiently smooth in the sense of Liapunov. The steady-state heat conduction in an isotropic medium is described by the following Poisson equation, namely

$$\nabla^2 u(x,y) = f(x,y), (x,y) \in \Omega, \tag{1}$$

where $u(x,y)$ is the potential field and $f(x,y)$ is the heat source term. The heat flux $q(x,y)$ through the boundary $\Gamma$ is given by

$$q(x,y) = \frac{\partial u(x,y)}{\partial \boldsymbol{n}} = \bar{q}(x,y), (x,y) \in \Gamma, \tag{2}$$

where $\boldsymbol{n}$ presents the outward unit vector normal to the boundary $\Gamma$, and the barred quantities indicate the given values on the boundary.

Assume that the heat source term $f(x,y)$ is unknown and both the temperature and heat flux can be measured on an accessible part of the boundary $\Gamma_1 \in \Gamma$, i.e.,

$$u(x,y) = \bar{u}(x,y), q(x,y) = \bar{q}(x,y), (x,y) \in \Gamma_1. \tag{3}$$

It should be noted that the inverse problem of recovering a heat source, in generally, does not admit a unique solution [29]. A minimum-norm solution to this problem is usually that of practical interest according to the so-called 'principle of parsimony', which states that, from the multitude of solutions to the inverse problem, the one that reveals the least amount of details or information should be selected. This has been previously exploited by Farcas et al. [32] to explain their results obtained using the Tikhonov regularization method. It turns out that the minimum-norm solution should satisfy the Laplace equation [33]. In order to illustrate the facility of the proposed scheme for incorporating different *a priori* assumptions, we consider the following two formulations of the problem:

**Formulation 1.** The heat source is harmonic in $\Omega$, i.e. the source satisfies the homogeneous Laplace equation $\nabla^2 f(x,y) = 0$. Applying the operator $\nabla^2$ to both side of Eq. (1) gives

$$\nabla^4 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, (x,y) \in \Omega, \tag{4}$$

$$u(x,y) = \bar{u}(x,y), (x,y) \in \Gamma_1, \tag{5}$$

$$q(x,y) = \bar{q}(x,y), (x,y) \in \Gamma_1. \tag{6}$$

**Formulation 2.** The heat source satisfies the homogeneous modified Helmholtz equation $(\nabla^2 - \lambda^2)f(x,y) = 0$, with the wave number $\lambda$ is known. Applying the operator $(\nabla^2 - \lambda^2)$ to both sides of Eq. (1) gives

$$\begin{aligned}(\nabla^2 - \lambda^2)\nabla^2 u &= \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} - \lambda^2 \frac{\partial^2 u}{\partial x^2} - \lambda^2 \frac{\partial^2 u}{\partial y^2} \\ &= 0, (x,y) \in \Omega,\end{aligned} \tag{7}$$

$$u(x,y) = \bar{u}(x,y), (x,y) \in \Gamma_1, \tag{8}$$

$$q(x,y) = \bar{q}(x,y), (x,y) \in \Gamma_1. \tag{9}$$

Although other formulations of the problem may be possible, in this study we investigate only the aforementioned two cases of the inverse heat source problem. Other formulations of the problems can be solved in a similar way. The problem is now transformed to the calculation of a fourth-order partial differential equation which can be solved conveniently and stably by using the GFDM, which will be presented in the following section.

## 3. Numerical algorithms

### 3.1. GFDM for fourth-order partial differential equations

Without loss of generality, let us consider a problem governed by the following forth-order partial differential equation

$$\begin{aligned}&a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + a_3 \frac{\partial^2 u}{\partial x^2} + a_4 \frac{\partial^2 u}{\partial y^2} + a_5 \frac{\partial^2 u}{\partial x \partial y} + a_6 \frac{\partial^3 u}{\partial x^3} + a_7 \frac{\partial^3 u}{\partial y^3} \\ &\quad + a_8 \frac{\partial^3 u}{\partial x^2 \partial y} + a_9 \frac{\partial^3 u}{\partial x \partial y^2} + a_{10} \frac{\partial^4 u}{\partial x^4} + a_{11} \frac{\partial^4 u}{\partial y^4} + a_{12} \frac{\partial^4 u}{\partial x^3 \partial y} \\ &\quad + a_{13} \frac{\partial^4 u}{\partial x^2 \partial y^2} + a_{14} \frac{\partial^4 u}{\partial x \partial y^3} = g(x,y).\end{aligned} \tag{10}$$

In the GFDM, by utilizing the Taylor series expansions and weighted least-squares method, the derivatives of unknown variables can be expressed as linear combinations of function values at its neighboring nodes. First of all, an irregular cloud of points