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Closed-form analytical solutions of transient heat conduction in hollow composite cylinders with any number of layers

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ABSTRACT

Conventional analytical methods for transient heat conduction solutions, due to complicated derivations and formidable calculations, have been limited to composite bodies with two or three layers. Developed in this work is a new analytical solution method for transient heat conduction in hollow composite cylinders with an arbitrary number of layers and subject to general boundary conditions. In this method, a distributed transfer function formulation gives an s-domain solution of a composite cylinder and inverse Laplace transform of the s-domain solution via a newly derived residue formula yields the transient solution of the cylinder in an explicit and closed form. Unlike conventional analytical methods, the proposed method does not require different derivations for different cylinder configurations (such as number of layers, boundary conditions and thermal resistance at layer interfaces). Because it only involves two-by-two matrices in the solution process, the proposed method is highly efficient in computation, as demonstrated in numerical examples.

1. Introduction

Hollow multilayer composite cylinders are found in a variety of engineering applications, including turbines, rockets, spacecraft, heat exchangers, fuel cells, electrochemical reactors, and pipelines in certain industrial processes. In many such applications, it is essentially important to acquire detailed knowledge on temperature and heat flow through a cylindrical composite media, which eventually is boiled down to a classic problem of transient (unsteady) heat conduction. For this kind of problems, numerical methods, such as finite difference methods and finite element methods, are commonly used. Although numerical methods are popular, analytical methods are still desirable for accuracy and efficiency in computation and for capability of providing deep physical insight. Additionally, analytical methods can serve as a benchmark for validation of numerical solutions. This work is about a new analytical method for transient heat conduction solutions of hollow composite cylinders with any number of layers and subject to general boundary conditions.

In the literature, various analytical methods are available for transient heat conduction problems, such as orthogonal and quasi-orthogonal expansion techniques [1–3], Laplace transform method [4–6], the line heat-source method [7,8], the method of separation of variables [9], Green's function method [10,11], finite integral transform technique [12], and their combinations [13]. In theory, analytical methods should be applicable to general multilayer bodies. In practice, however, analytical methods have been limited to composite media with two or three layers [14,15]. Issues like complicated expressions, formidable derivations, and numerical errors in inverse Laplace transform, just to name a few, prevent conventional analytical methods from obtaining transient solutions for composite bodies with many layers.

It is worth noting that continued effort has been made recently to advance Laplace transform method for analytical solutions of transient heat conduction in composite cylinders. In Ref. [16], an s-domain formulation is established for unsteady conductive heat transfer in cylindrical orthotropic composite laminates with general boundary conditions. Because exact inverse Laplace transform is difficult to obtain, the complex integral involved in Laplace inversion is evaluated through use of meromorphic functions. In Ref. [17], through application of the Cauchy's theorem to the integral in inverse Laplace transform, explicit and closed-form transient solutions of heat conduction are obtained for a two-layer composite cylinder. The solution formulas, while elegantly derived, are expected to encounter increased complexity if more layers are considered. These new results indicate that Laplace transform method is potentially useful for analytical solutions of transient heat conduction in composite cylinders with many layers.

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On the other hand, in the areas of structural dynamics and solid mechanics, an analytical method, called the distributed transfer function method (DTFM), has been developed for modeling and analysis of deformable (flexible) multibody systems [18–23]. The DTFM treats an assembly of multiple bodies in a symbolic manner, does not require tedious derivations, and is capable of delivering analytical solutions with accuracy and efficiency. It is with these advantages that the DTFM is applied to obtain closed-form solutions of transient heat conduction in one-dimensional multilayer slabs [24].

Extending the concepts of the DTFM in Ref. [24], this effort is aimed to obtain closed-form analytical solutions for transient heat conduction in hollow composite cylinders. A closed-form analytical solution herein refers to one that is given by an infinite series with each term presented in an exact mathematical expression with a finite number of terms. The problem in consideration is one-dimensional, with temperature distribution in the radial direction. In the development, a distributed transfer function formulation and a formula for transfer function residues are derived, which eventually yields transient heat conduction solutions in explicit and closed form. The thrust of the proposed method lies in that it is applicable to composite cylinders of an arbitrary number of layers and subject to general boundary conditions. To the best of the authors' knowledge, this analytical method is new in the field of heat conduction.

The remainder of this article is arranged as follows. The governing equations of the conductive heat transfer problem are presented in Section 2 and the equivalent spatial state equations in the s domain and a distributed transfer function formulation are derived in Section 3. With the state equations and transfer function formulation, the eigenvalues and steady-state response of the heat conduction problem are obtained in Sections 4 and 5. Section 6, a formula for exact transfer function residues is first devised, and the inverse Laplace transform of the s-domain solution given in Section 3 is then determined with the transfer function residues, yielding the closed-form transient solution subject to general initial condition, boundary excitations and internal heat generation. In Section 7, the transient analysis is extended to composite cylinders with thermal resistance at layer interfaces and fast-converging transient solutions are derived in the case of temperature jumps at boundaries. In Section 8, the proposed DTFM is illustrated on two composite cylinders in simulation, one with two layers and the other with seven layers. Finally, the major outcomes from this effort are summarized in Section 9.

2. Problem description

The composite media in consideration is an n-layer hollow cylinder, as shown in Fig. 1, where $0 < r_0 \leqslant r \leqslant r_n$, r_{i-1} and r_i mark the inner and outer radii of the i th layer, and r_0 and r_n indicate the inner and outer surfaces of the cylinder. Let the temperature distribution through each layer be axisymmetric. The governing equation of heat conduction in the i th layer then is

$$\frac{\partial^2 T_i(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T_i(r,t)}{\partial r} + \frac{g_i(r,t)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i(r,t)}{\partial t}, \quad r_{i-1} \leqslant r \leqslant r_i \tag{1}$$

where $T_i(r,t), \alpha_i, k_i$ are the temperature, thermal diffusivity, and thermal conductivity of the layer, respectively; and $g_i(r,t)$ represents heat generation within the layer. The boundary conditions on the inner and outer surfaces of the cylinder are of the general form

$$\begin{aligned} &at\ r = r_0: \quad a_1\frac{\partial T_1(r,t)}{\partial r} + a_0 T_1(r,t) = \gamma_{in}(t) \\ &at\ r = r_n: \quad b_1\frac{\partial T_n(r,t)}{\partial r} + b_0 T_n(r,t) = \gamma_o(t) \end{aligned} \tag{2}$$

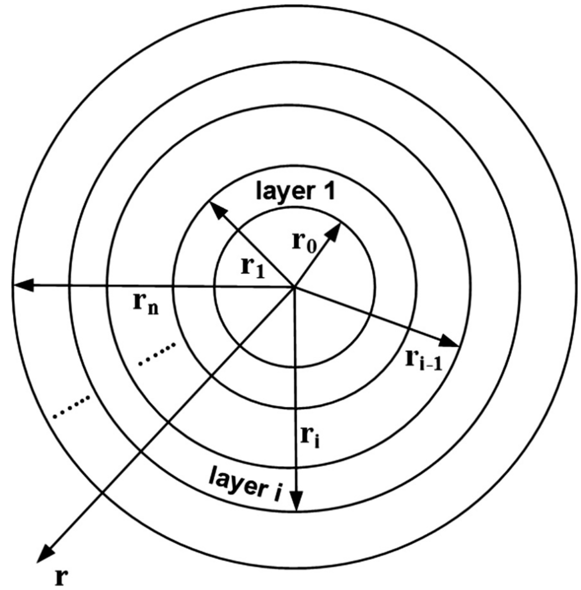

Fig. 1. A hollow n-layer composite cylinder.

where a_0, a_1, b_0, and b_1 are constants that are properly selected for different types of boundary conditions, with $a_0^2 + a_1^2 \neq 0$ and $b_0^2 + b_1^2 \neq 0$; and $\gamma_{in}(t)$ and $\gamma_o(t)$ are the external disturbances (prescribed temperature and/or heat flux) on the inner and outer surfaces of the cylinder. Assume that all the layers are in perfect thermal contact. (Thermal resistance at layer interfaces will be discussed in Section 7.1.) The matching conditions describing the continuity of temperature and heat flux at the layer interfaces are given by

$$\begin{aligned} &T_i(r_i,t) = T_{i+1}(r_i,t) \\ &-k_i\frac{\partial T_i(r_i,t)}{\partial r} = -k_{i+1}\frac{\partial T_{i+1}(r_i,t)}{\partial r} \end{aligned} \tag{3}$$

for $i = 2, 3, \ldots, n-1$. Additionally, the initial condition is

$$T_i(r,0) = \theta_i(r), \quad r_{i-1} \leqslant r \leqslant r_i \tag{4}$$

where $\theta_i(r)$ is a given initial temperature distribution through the ith layer.

Equations (1)–(4) form a boundary-initial value problem for transient (unsteady) heat conduction in the cylindrical composite with n layers. The objective of this study is to develop an analytical method for closed-form solutions of the boundary-initial value problem.

3. Distributed transfer function formulation

In this section, through the establishment of a spatial state representation and a distributed transfer function formulation, an s-domain solution of the cylindrical composite is obtained. The time-domain solution of the boundary-initial value problem shall be determined via a new method for inverse Laplace of the s-domain solution in Section 6.

3.1. Spatial state representation

Performing Laplace transformation of Eqs. (1)–(3) with respect to time t gives the governing equations within the layers

$$\frac{\partial^2 \overline{T}_i(r,s)}{\partial r^2} = -\frac{1}{r}\frac{\partial \overline{T}_i(r,s)}{\partial r} - \frac{\overline{g}_i(r,s)}{k_i} + \frac{s}{\alpha_i}\overline{T}_i(r,s) - \frac{1}{\alpha_i}\theta_i(r), \quad i = 1, 2, \ldots, n \tag{5}$$