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Large time steps in the explicit formulation of transient heat transfer

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ABSTRACT

In this paper, a generalized formulation of stiffness and mass using modified integration rules (MIR) with flexible integration points is developed to improve the stability of transient heat transfer problems. With adjustment of integration points in the stiffness, the softening or stiffening properties of discretized model for heat transfer problems can be altered. In addition, it is found that the integration points in the mass have a great effect on the critical time step for the explicit formulation of transient heat transfer problems. With a proper selection of integration points in the mass, a much larger time step can be applied in the analysis of transient heat transfer problems. Furthermore, it is observed that the final steady solutions of transient heat transfer problems are identical regardless of locations of integration points in the mass model. Numerical experiments including 2D heat transfer problems with different boundary conditions including heat conduction, heat convection and radiation are studied to verify the properties of flexible integration points in the stiffness and mass.

1. Introduction

In the process of heat transfer problems of practical engineering, the efficient simulation and precise modelling are extremely important. However, the analytical solutions are only available for linear problems and mostly constrained to simple geometrical domains. In addition, the experimental approach could be expensive in most of engineering problems. For these reasons, many type of numerical methods have been developed in the past decades such as Boundary Element Method (BEM) [1], Finite Difference Methods (FDM) [2,3], Finite Element Methods (FEM) [4–6] and meshfree method [7], etc. Currently, the FEM is the probably the most popular numerical method due to their attractive features in handling the problems with complicated geometry and flexibility for many types of problems [8].

In general, two types of integration methods including explicit [9–11] and implicit methods [12,13] are widely used in the simulation of transient heat transfer problems. The key feature of the explicit time-domain technique is that the evaluation of the field solutions is processed based on the previous time step. Thus, it is unnecessary to solve a system of equations involving any iterative algorithm per time step. Although the computational effort per time step in the explicit method is less than that using the implicit method, the conditional stability is considered as the main shortcoming of explicit methods. As is known to all, a very small time step must be used to ensure a stable solution in the explicit method. That leads to a large number of time steps in the simulation of transient heat transfer problems. On the contrary, the implicit method plays a complementary role in the time integration. In the implementation of the implicit method, the computation of the field solution at each time step must be solved iteratively before proceeding to next time step. As the implicit method is unconditionally stable for linear problems [12–15], a large time step could be applied in the modelling.

In order to improve the computational efficiency of explicit method, a lot of research effort following different directions has been made in the past [9,11,14–18]. Recently, modified integration rules (MIR) in the computation of the mass and stiffness were developed by Guddati and Yue [19,20] for acoustic problems using quadrilateral mesh. Following this, with employment of triangular elements in 2D and tetrahedral elements in 3D, the mass-redistributed finite element method (MR-FEM) with a parameter r controlling the distribution of mass was developed in our past work [21–23]. Our previous work has discovered an important fact that the stability of dynamic quasi-harmonic system largely relies on the maximum eigenfrequency of mass system [23,24]. With re-distribution of mass in the mass matrix using triangular or

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tetrahedral elements, the numerical stability of transient quasi-harmonic model can be improved significantly. This can be easily done by shifting the integration point locations when computing the entries of the mass matrix, while ensuring the mass conservation.

In general, the lower-order quadrilateral elements are very popular in FEM framework, and such elements are widely used in practical application due to its accuracy and efficiency [8]. In this work, aligned with the idea of MIR and the MR-FEM, the stiffness and mass using MIR with flexible integration points are formulated, which is aimed to improve the computational efficiency of dynamic heat transfer problems. With alternation of the location of integration points in the stiffness and mass models, it is found that the softening and stiffening effect of the discretized model can be adjusted. In addition, it is proved that the numerical stability of transient heat transfer model can be improved significantly based on the theoretical analysis and numerical examples. More importantly, the steady solutions are always identical (not by chance) regardless of integration point in the mass model, which gives us the flexibility to adopt much larger time steps in the explicit formulation of transient heat transfer problems without losing the accuracy. Based on the quantitative study, it is found that the numerical stability of discretized model is proportional to r value as $r > 0$ in the mass model. While the critical time step in the transient heat transfer model decreases with increasing integration point p in the stiffness model. The successful development of robust, efficient and accurate explicit algorithms has opened an important window in the simulation of general transient heat transfer problems via manipulating the integration points directly.

The paper is organized as follows: Section 2 briefly describes the standard FEM to solve the transient heat transfer problems, and the mathematical formulations of stiffness and mass using MIR with flexible integration points are also presented in Section 2. Next, the stability of general transient heat transfer problems is investigated in Section 3. A number of examples are studied in detail to evaluate the performance of stiffness and mass with flexible integration points in Section 4. Finally the conclusions from the theoretical analysis and numerical results are made in Section 5.

2. Formulation of stiffness and mass models using MIR with flexible integration points

2.1. Discretization of governing equation of dynamic heat transfer using FEM

For more effective discussion, the standard FEM using quadrilateral elements for transient heat transfer problem is first briefed.

The governing equation for transient heat transfer problem in a 2D domain Ω is expressed as follows:

$$\frac{\partial}{\partial x}\left(\kappa_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa_y\frac{\partial T}{\partial y}\right)+Q(x,y,t)=\rho c\frac{\partial T(x,y,t)}{\partial t} \tag{1}$$

where c is the specific heat capacity of medium, Q is the source term representing internal heat generation, ρ is the density of medium, T is unknown temperature, t is the time, and κ is the thermal conductivity.

The boundary conditions for transient heat transfer problems are given as follows:

Essential boundary condition:

$$T=T_e \quad \text{on } \Gamma_1 \tag{2}$$

Convection boundary condition:

$$-\kappa_x\frac{\partial T}{\partial x}n_x-\kappa_y\frac{\partial T}{\partial y}n_y=h(T-T_a) \quad \text{on } \Gamma_2 \tag{3}$$

where h is the convective heat transfer coefficient, and n is the unit normal vector.

Heat flux boundary condition

$$-\kappa_x\frac{\partial T}{\partial x}n_x-\kappa_y\frac{\partial T}{\partial y}n_y=q \quad \text{on } \Gamma_3 \tag{4}$$

where q is the prescribed heat flux.

Adiabatic boundary condition

$$\kappa_x\frac{\partial T}{\partial x}n_x+\kappa_y\frac{\partial T}{\partial y}n_y=0 \quad \text{on } \Gamma_4 \tag{5}$$

Radiation boundary condition

$$-\kappa_x\frac{\partial T}{\partial x}n_x-\kappa_y\frac{\partial T}{\partial y}n_y=\sigma\varepsilon\left(T^4-T_R^4\right) \quad \text{on } \Gamma_5 \tag{6}$$

where σ is the Stefan Boltzmann constant for radiation (5.67×10^{-8} W m^{-2} K^{-4}), and ε is the emissivity taken as 1 unit in this work.

Based on the standard FEM weak formulation, Eq. (1) can be finally written in the following matrix form:

$$\mathbf{F}^t=[\mathbf{M}]\dot{\mathbf{T}}^t+[\mathbf{K}+\mathbf{C}]\mathbf{T}^t \tag{7}$$

where

$$\mathbf{K}=\boldsymbol{\kappa}\int_\Omega \nabla\mathbf{N}^{\mathrm{T}}\nabla\mathbf{N}\mathrm{d}\Omega \quad \boldsymbol{\kappa}=\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix} \quad \text{Stiffness matrix} \tag{8}$$

$$\mathbf{F}=\int_\Omega \mathbf{N}^TQ\,d\Omega-\int_{\Gamma_2}\mathbf{N}^Tq\,d\Gamma_2+\int_{\Gamma_3}hT_a\mathbf{N}^T\,d\Gamma_3+\int_{\Gamma_5}\sigma\varepsilon(T_R^4-T^4)\mathbf{N}^T\,d\Gamma_5 \quad \text{Equivalent force matrix} \tag{9}$$

$$\mathbf{C}=\int_{\Gamma_3}h\mathbf{N}^T\mathbf{N}\,\mathrm{d}\Gamma_3 \quad \text{Equivalent damping matrix} \tag{10}$$

$$\mathbf{M}=\rho c\int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Omega \quad \text{Mass matrix} \tag{11}$$

where $\mathbf{N}$ is shape function created by standard FEM.

In the explicit formulation, the equilibrium equation at time t can be written as follows:

$$\mathbf{F}^t=\mathbf{M}\frac{\mathbf{T}^{t+\Delta t}-\mathbf{T}^t}{\Delta t}+(\mathbf{K}+\mathbf{C})\mathbf{T}^t \tag{12}$$

Re-arranging the terms leads to

$$\mathbf{M}\mathbf{T}^{t+\Delta t}=\mathbf{F}^t\Delta t-(\mathbf{K}+\mathbf{C})\mathbf{T}^t\Delta t+\mathbf{M}\mathbf{T}^t \tag{13}$$

2.2. Formulations of stiffness and mass using MIR with flexible integration points

In the FEM framework, the isoparametric elements and Gauss integration are usually adopted to calculate the entries of stiffness matrix $\mathbf{K}_{IJ}$ as follows

$$\begin{aligned}\mathbf{K}_{IJ}&=\int_\Omega \mathbf{B}_I^{\mathrm{T}}(\mathbf{x})\boldsymbol{\kappa}\mathbf{B}_J(\mathbf{x})\mathrm{d}\Omega\\&=\int_{-1}^1\int_{-1}^1\mathbf{B}_I^{\mathrm{T}}(x(\xi,\eta))\boldsymbol{\kappa}\mathbf{B}_J(x(\xi,\eta))|J(\xi,\eta)|\mathrm{d}\xi\mathrm{d}\eta\\&=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\mathbf{B}_I^{\mathrm{T}}(\mathbf{x}(\xi_i,\eta_i))\boldsymbol{\kappa}\mathbf{B}_J\times(\mathbf{x}(\xi_i,\eta_i))|\mathbf{J}(\xi_i,\eta_i)|W_i\overline{W}_j\end{aligned} \tag{14}$$

$$\mathbf{B}_I(\mathbf{x})=\nabla_sN_I(\mathbf{x}) \tag{15}$$

where N_1 and N_2 are the number of Gauss integration points in the ξ and η axes, respectively. In addition, ξ_i, η_j are the integration points and W_i and W_j are weighting coefficients.