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Non-Boussinesq stability analysis of natural-convection gaseous flow on inclined hot plates

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A B S T R A C T

The buoyancy-driven boundary-layer flow that develops over a semi-infinite inclined hot plate is known to become unstable at a finite distance from the leading edge, characterized by a critical value of the Grashof number Gr based on the local boundary-layer thickness. The nature of the resulting instability depends on the inclination angle ϕ, measured from the vertical direction. For values of ϕ below a critical value ϕ_c the instability is characterized by the appearance of spanwise traveling waves, whereas for $\phi > \phi_c$ the bifurcated flow displays Görtler-like streamwise vortices. The Boussinesq approximation, employed in previous linear stability analyses, ceases to be valid for gaseous flow when the wall-to-ambient temperature ratio Θ_w is not close to unity. The corresponding non-Boussinesq analysis is presented here, accounting also for the variation with temperature of the different transport properties. A temporal stability analysis including nonparallel effects of the base flow is used to determine curves of neutral stability, which are then employed to delineate the dependences of the critical Grashof number and of its associated wave length on the inclination angle ϕ and on the temperature ratio Θ_w for the two instability modes, giving quantitative information of interest for configurations with $\Theta_w - 1 \sim 1$. The analysis provides in particular the predicted dependence of the crossover inclination angle ϕ_c on Θ_w, indicating that for gaseous flow with $\Theta_w - 1 \sim 1$ spanwise traveling waves are predominant over a range of inclination angles $0 \leqslant \phi \leqslant \phi_c$ that is significantly wider than that predicted in the Boussinesq approximation.

1. Introduction

A semi-infinite inclined hot plate placed in a quiescent air atmosphere is known to induce near its surface a free-convection flow as a result of the action of buoyancy forces on the heated gas. The structure of the resulting boundary layer away from the plate edge exhibits at leading order a self-similar structure, as first noted in the experimental study of Schmidt and Beckmann [1]. This boundary layer is known to become unstable to small disturbances at a certain distance measured from the leading edge of the plate [2]. The character of the observed instability depends on the inclination angle ϕ, measured from the vertical direction. Thus, for values of ϕ above a critical value ϕ_c, including in particular horizontal and nearly horizontal plates, the instability develops in the form of stationary counter-rotating vortex rolls that are oriented in the streamwise direction. These are similar to those characterizing the Görtler instability of boundary-layer flow along a concave wall, driven by centrifugal forces, with the wall-normal component of the buoyancy force being the driving mechanism for free-convection flow. As the inclination angle ϕ is decreased, this wall-normal buoyancy component loses importance and, below a certain crossover angle ϕ_c, the character of the observed instabilities changes to Tollmien-Schlichting-like traveling waves driven by shear. Following existing terminology [3,4], in the following the stability mode involving streamwise stationary vortices will be termed *vortex instability*, whereas that involving traveling waves will be termed *wave instability*.

Sparrow & Husar [2] were the first to identify both modes experimentally, and to show that their prevalence depends on the inclination of the heated surface. The crossover angle was determined by Lloyd & Sparrow [5] to lie between $14^\circ < \phi_c < 17^\circ$. Other experiments carried out later agree generally with these findings [6–11].

Apart from the inclination angle ϕ, the buoyancy-induced flow over a semi-infinite flat plate at constant temperature depends on the Prandtl number Pr of the fluid and on the ratio $\Theta_w = T^*_w/T^*_\infty$ of the wall temperature to the ambient temperature. All previous

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theoretical efforts aimed at quantifying the critical condition at the onset of the vortex and wave instabilities were performed in the Boussinesq approximation [3,4,12–20], which is only justified in gaseous flow when the wall-to-ambient relative temperature difference $(\Theta_w - 1)$ is small. Most of these studies employ linear local stability theory—a normal mode analysis—to determine, for fixed values of Pr and ϕ, the critical boundary-layer thickness δ_o^*, measured in dimensionless form through a local Grashof number, above which small perturbations, either of vortex type with associated spanwise wave number l^*, or of wave type with streamwise wave number k^*, are amplified. In this manner, a unique neutral curve in the Grashof – wave number plane can be delineated for each mode. The mode with the lowest corresponding critical Grashof number for all wave numbers would be the one that prevails in an experiment, and the value of that Grashof number would give the local boundary-layer thickness—and therefore the distance x^* to the plate edge—at which the instability first develops.

Conventionally, in a local stability analysis the base flow is assumed to be strictly parallel. That assumption must be reconsidered in the analysis of slowly varying slender flows, such as the present boundary layer, for which the order of magnitude of some of the terms in the stability equations, involving the transverse velocity component and the streamwise variation of the flow, is comparable to that of the viscous terms, and must be correspondingly taken into account. This so-called locally nonparallel approach was already adopted by Haaland & Sparrow in their temporal stability analyses of the vortex [3] and wave [4] instability modes. The resulting critical Grashof numbers were seen to differ by several orders of magnitude from those obtained with a strictly parallel analysis, thereby underlining the importance of the nonparallel terms. The problem was re-examined by a series of authors, adopting small variations of this approach, either in a temporal [12,13,15–19] or a spatial [14] linear–stability framework. The analysis can be extended to describe finite-amplitude vortex rolls and secondary bifurcations by retaining selected nonlinear terms in the description, as done by Chen et al. [19]. Recently, instabilities in transient cooling of inclined surfaces and cavities have been studied numerically [21,22].

The accuracy of the computations mentioned above deteriorates in the presence of order–unity deviations of the wall temperature from the ambient temperature, when the use of the Boussinesq approximation is no longer justified. Although non-Boussinesq effects have been taken into account in analyses of the boundary-layer structure for flow over a heated plate [23–25], these effects have never been considered in connection with the associated stability problem. The objective of the present work is to revisit the classical work of Haaland & Sparrow [3,4], including the influence of the wall-to-ambient temperature ratio Θ_w for cases with $\Theta_w - 1 \sim 1$. In particular, a linear temporal modal stability analysis will be performed to investigate both the vortex and the wave modes, with account taken of nonparallel effects associated with the slow downstream evolution of the base flow. The effect of Θ_w on the neutral stability curves will be assessed, along with the dependence of the crossover angle defining the transition between the two types of instability.

The paper is structured as follows. The governing equations and boundary conditions for the base flow and for the linear stability analysis are given in Section 2. The vortex mode is studied in Section 3, followed in Section 4 by the analysis of the wave instability. The predictions of the critical conditions for the two modes are compared in Section 5 to delineate the boundary that defines the regions of prevalence of each mode on the parametric plane $\phi - \Theta_w$. Finally, concluding remarks are offered in Section 6.

2. Problem formulation

The problem considered here, shown schematically in Fig. 1, involves the flow induced by buoyancy near the surface of a semi-infinite inclined plate whose temperature is held at a constant value, T_w^*, higher than the ambient temperature T_∞^* found in the surrounding quiescent air atmosphere. The associated velocities are negligibly small compared to the sound speed, so that the conservation equations can be written in the low-Mach number approximation

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \boldsymbol{v}^*) = 0, \tag{1}$$

$$\rho^* \frac{\partial \boldsymbol{v}^*}{\partial t^*} + \rho^* \boldsymbol{v}^* \cdot \nabla^* \boldsymbol{v}^* = -\nabla^* p^* + (\rho^* - \rho_\infty^*)\boldsymbol{g}^* + \nabla^* \cdot \left[\mu^*(\nabla^* \boldsymbol{v}^* + \nabla^* \boldsymbol{v}^{*T})\right], \tag{2}$$

$$\rho^* \frac{\partial T^*}{\partial t^*} + \rho^* \boldsymbol{v}^* \cdot \nabla^* T^* = \frac{1}{Pr} \nabla^* \cdot (\mu^* \nabla^* T^*), \tag{3}$$

where ρ^*, $\boldsymbol{v}^*$, and T^* represent the density, velocity, and temperature of the gas; dimensional quantities are indicated everywhere in the text with an asterisk $(*)$. In the momentum Eq. (2), p^* represents the sum of the pressure difference from the ambient hydrostatic distribution and the isotropic component of the stress tensor. Cartesian coordinates are used in the description, including the streamwise distance measured along the plate from the leading edge x^*, the transverse distance from the surface of the plate y^*, and the spanwise coordinate z^*, with corresponding velocity components $\boldsymbol{v}^* = (u^*, v^*, w^*)$. The inclination angle ϕ is measured from the vertical, so that the gravity vector is $\boldsymbol{g}^* = -g^* \cos\phi \boldsymbol{e_x} - g^* \sin\phi \boldsymbol{e_y}$.

The above equations must be supplemented with the equation of state

$$\frac{\rho^*}{\rho_\infty^*} = \frac{T_\infty^*}{T^*} \tag{4}$$

and with the presumed power law

$$\frac{\mu^*}{\mu_\infty^*} = \frac{\kappa^*}{\kappa_\infty^*} = \left(\frac{T^*}{T_\infty^*}\right)^\sigma \tag{5}$$

for the temperature dependence of the viscosity and thermal conductivity, with the subscript ∞ denoting properties in the unperturbed ambient air. The constant values $Pr = 0.7$ and $\sigma = 2/3$, corresponding to air, will be used below for the Prandtl number $Pr = c_p^* \mu_\infty^* / \kappa_\infty^*$ in (3) and for the exponent σ in (5). Eqs. (1)–(3) must be integrated with the boundary conditions

$$\begin{cases} u^* = v^* = w^* = T^* - T_w^* = 0 & \text{at } y^* = 0 \text{ for } x^* > 0 \\ u^* = v^* = w^* = T^* - T_\infty^* = p^* = 0 & \text{as } (x^{*2} + y^{*2}) \to \infty \text{ for } y^* \neq 0, x^* > 0. \end{cases} \tag{6}$$

For plates that are not nearly horizontal, such that $\pi/2 - \phi$ is not small, the flow is driven by the direct acceleration associated with the gravity component parallel to the plate $g^* \cos\phi$. Near the leading edge of the plate there exists a nonslender Navier–Stokes region of characteristic size $[{\nu_\infty^*}^2/(g^* \cos\phi)]^{1/3}$ where the velocity components are of order $(\nu_\infty^* g^* \cos\phi)^{1/3}$, with $\nu_\infty^* = \mu_\infty^*/\rho_\infty^*$ denoting the ambient kinematic viscosity, such that the local Reynolds number there is of order unity. Outside this Navier–Stokes region the flow–field structure includes a boundary-layer region of characteristic thickness $[({\nu_\infty^*}^2 x^*)/(g^* \cos\phi)]^{1/4}$ and characteristic streamwise velocity $(g^* \cos\phi x^*)^{1/2}$, surrounded by an outer region of slow irrotational motion driven by the boundary-layer entrainment.