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A time-domain FEM approach based on implicit Green's functions for the dynamic analysis of porous media

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ABSTRACT

This paper describes an original time-domain formulation to analyse saturated porous media. Standard finite element procedures are employed to numerically discretize the spatial domain of the model and a time-marching scheme based on the mechanical Green's function of the problem is considered. The Green's function matrices are implicitly and numerically evaluated, taking into account the Newmark method. The present methodology allows the system of equations of the solid and fluid phase to be treated separately, providing a more efficient and accurate solution procedure (smaller, simpler and better conditioned systems of equations are analysed). At the end of the paper, numerical examples are presented, illustrating the potentialities of the new approach.

1. Introduction

For many everyday engineering problems, such as earthquake engineering, soil–structure interaction, biomechanics, seismic wave scattering, etc., dynamic porous media analysis is necessary and over simplified theoretical models (e.g., pure elastodynamic theory, etc.) may only represent a very crude approximation. Nowadays, several numerical approaches, especially those considering finite element procedures, are available to analyse complex dynamic porous media (see [1–5], for instance) and most of these approaches are based on the pioneering work of Biot [6–9] (for a complete overview of the porous media theory evolution, Ref. [10] is recommended).

The present work is focused on the numerical modelling of saturated soils (i.e., soils that are composed of a solid phase with voids filled with water) and it is based on the $\mathbf{u}-p$ formulation, as presented by Zienkiewicz et al. [11,12]. As commonly reported, the $\mathbf{u}-p$ formulation is a very attractive approach because of both its performance and simplicity; the variables of the model are, in this case, the displacements of the soil skeleton ($\mathbf{u}$) and the pressures of the pore fluid (p).

In this paper, the pore-pressure field is calculated taking into account usual time-integration techniques, and the displacements of the model (and their time derivatives) are computed based on implicit Green's function matrices. This time-integration technique was introduced by Soares [13] and Soares and Mansur [14,15], considering dynamic applications modelled by the finite element method. Subsequently, the methodology was also employed to develop alternative finite element/boundary element coupling procedures [16,17], as well as alternative boundary element time-marching schemes [18].

In the implicit Green approach, the displacements and velocities of the soil skeleton are evaluated considering a recurrence relationship that employs the time-domain Green's matrices of the dynamic problem. These Green's matrices are evaluated implicitly and numerically, taking into account standard time-integration algorithms. The present work focuses on the Newmark method to implicitly calculate the Green's matrices and, as has been demonstrated [13,14], if the trapezoidal rule is considered, the Green approach becomes second order accurate and unconditionally stable.

The expressions that arise from the implicit Green approach are quite simple and effective, allowing the coupled system of equations of the porous medium to be properly modified. This renders two smaller and simpler to deal with systems of equations: one related to the solid phase and another related to the fluid phase. As a consequence, a more efficient and better-conditioned overall methodology is obtained.

Considering standard finite element analysis, in the limit of zero compressibility of water and soil grains and zero permeability, the functions used to interpolate displacements and pressures must fulfil either the Babuska and Brezzi [19,20] conditions or the simpler patch test proposed by Zienkiewicz et al. [21]. These requirements exclude the use of elements with equal order interpolation for pressures and displacements, for which spurious oscillations

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may appear. Several works have been presented regarding this matter, as for instance the work of Huang and Zienkiewicz [22], which presents a new class of unconditionally stable staggered implicit–implicit time-stepping algorithms for coupled soil-pore fluid dynamic problems; Pastor et al. [23], which describes a stabilization technique that allows the use of both linear triangles or both bilinear quadrilaterals for displacements and pressures; Pastor et al. [24] and Li et al. [25], proposing a generalization of the fractional-step method and its modified version, respectively, etc. (Huang et al. [26] summarize the stabilization techniques that have been proposed in the literature to overcome volumetric locking for the incompressible or nearly incompressible soil dynamic behaviours). Alternative approaches, as those presented by Tchonkova et al. [27] (where a mixed least-squares method for solving Biot consolidation problems is developed) and by Zhang and Zhou [28] (where the numerical manifold method is presented), should also be highlighted.

As is reported in the text that follows, taking into account the present proposed formulation, no special procedures have to be considered when analysing incompressible and impermeable media (the same discretization may be considered for both solid and fluid phases, etc.), which also greatly contributes for the efficiency of the methodology (especially when adaptive meshes are to be considered).

In this paper, first (Section 2) the governing equations of the model are presented and briefly discussed. Next, the finite element modelling is considered and, in Section 3, the basic finite element vectors and matrices are presented (Section 3 also describes a standard finite element formulation to treat saturated pore-solids, which is employed later, providing numerical results for comparison). In Section 4, the implicit Green approach is discussed and the proposed formulation to analyse pore-dynamic media is presented. The stability analysis of the proposed methodology is carried out in Section 5. Finally, at the end of the paper (Section 6), two numerical applications are considered, illustrating the efficiency, accuracy and flexibility of the new technique.

2. Governing equations

The present work is focused on the $\mathbf{u}-p$ formulation, as presented by Zienkiewicz et al. [12]. The governing equations of the model are

$$\sigma_{ij,j} - \rho_m \ddot{u}_i + \rho_m b_i = 0, \tag{1}$$

$$\alpha\dot{\varepsilon}_{ii} - (k_{ij}p_{,j})_{,i} + (1/Q)\dot{p} + a = 0, \tag{2}$$

where Eq. (1) stands for the balance of momentum of the mixture and Eq. (2) is a combination of the balance of mass and momentum for the interstitial fluid.

In Eq. (1), σ_{ij} is the total Cauchy stress (usual indicial notation for Cartesian axes is considered); the effective stress is defined as $\sigma'_{ij} = \sigma_{ij} + \alpha\delta_{ij}p$, where α accounts for slight strain changes, p stands for interstitial fluid pore-pressure and δ_{ij} represents the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$). Further on in Eq. (1), u_i stands for the solid matrix displacement and b_i for the body force distribution. Inferior commas and overdots indicate partial space ($u_{j,i} = \partial u_j/\partial x_i$) and time ($\dot{u}_i = \partial u_i/\partial t$) derivatives, respectively. $\rho_m = \mu\rho_f + (1-\mu)\rho_s$ stands for the mass density of the mixture, where ρ_s and ρ_f are the mass density of the solid and fluid phase, respectively, and μ is the porosity of the medium. In Eq. (2), ε_{ij} represents the strain tensor and k_{ij} defines the permeability coefficients, according to the D'Arcy seepage law. a stands for domain source terms and the mixture parameter Q is defined by $(1/Q) = \mu/K_f + (\alpha - \mu)/K_s$, where the bulk moduli of the solid and fluid phase are represented by K_s and K_f, respectively.

Eqs. (1) and (2), accompanied by appropriate initial and boundary conditions, as well as proper constitutive laws, define the pore-dynamic model under consideration. In the next section, discretization techniques, taking into account the finite element method, are briefly discussed.

3. Standard finite element solution

Taking into account standard finite element techniques, after introducing spatial approximations (as described by $\mathbf{u}(\mathbf{x}, t) = \mathbf{N}_u\mathbf{U}(t)$ and $p(\mathbf{x}, t) = \mathbf{N}_p\mathbf{P}(t)$, where $\mathbf{N}$ stands for spatial interpolation matrices), the following system of equations can be obtained, regarding Eqs. (1) and (2) [2,3,12]:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \int_\Omega \mathbf{B}^T\boldsymbol{\sigma}'(t)\mathrm{d}\Omega - \mathbf{Q}\mathbf{P}(t) - \mathbf{f}_u(t) = \mathbf{0}, \tag{3}$$

$$\mathbf{Q}^T\dot{\mathbf{U}}(t) + \mathbf{H}\mathbf{P}(t) + \mathbf{S}\dot{\mathbf{P}}(t) - \mathbf{f}_p(t) = \mathbf{0}, \tag{4}$$

where $\mathbf{B}$ is the well-known strain matrix. The mass ($\mathbf{M}$), permeability ($\mathbf{H}$), compressibility ($\mathbf{S}$) and coupling ($\mathbf{Q}$) matrices are defined as follows:

$$\mathbf{M} = \int_\Omega \mathbf{N}_u^T\rho_m\mathbf{N}_u\mathrm{d}\Omega, \tag{5a}$$

$$\mathbf{H} = \int_\Omega \nabla\mathbf{N}_p^T\mathbf{k}\nabla\mathbf{N}_p\mathrm{d}\Omega, \tag{5b}$$

$$\mathbf{S} = \int_\Omega \mathbf{N}_p^T\frac{1}{Q}\mathbf{N}_p\mathrm{d}\Omega, \tag{5c}$$

$$\mathbf{Q} = \int_\Omega \mathbf{B}^T\alpha\mathbf{m}\mathbf{N}_p\mathrm{d}\Omega, \tag{5d}$$

where vector $\mathbf{m}$ (Eq. (5d)) plays the role of the Kronecker delta δ_{ij} and the entries of matrix $\mathbf{k}$ (Eq. (5b)) are defined by the permeability coefficients k_{ij}.

Vectors $\mathbf{f}_u(t)$ and $\mathbf{f}_p(t)$ (in Eqs. (3) and (4), respectively) account for prescribed traction ($\bar{\mathbf{t}}$) and flux ($\bar{\mathbf{q}}$) boundary conditions, as well as domain forces ($\mathbf{b}$) and sources ($\mathbf{a}$). They are defined as follows:

$$\mathbf{f}_u(t) = \int_{\Gamma_t} \mathbf{N}_u^T\bar{\mathbf{t}}(t)\mathrm{d}\Gamma + \int_\Omega \mathbf{N}_u^T\rho_m\mathbf{b}(t)\mathrm{d}\Omega, \tag{6a}$$

$$\mathbf{f}_p(t) = \int_{\Gamma_q} \mathbf{N}_p^T\bar{\mathbf{q}}(t)\mathrm{d}\Gamma + \int_\Omega \mathbf{N}_p^T\mathbf{a}(t)\mathrm{d}\Omega, \tag{6b}$$

where Ω and Γ stand for the domain and the boundary of the model, respectively ($\Gamma = \Gamma_u \cup \Gamma_t = \Gamma_p \cup \Gamma_q$).

Eqs. (3)–(6) describe finite element spatial discretization procedures. For time discretization, the following finite difference approximations may be considered (generalized Newmark method):

$$\dot{\mathbf{U}}^n = \dot{\mathbf{U}}^{n-1} + (\Delta t)\ddot{\mathbf{U}}^{n-1} + (\gamma_1\Delta t)\Delta\ddot{\mathbf{U}}, \tag{7a}$$

$$\mathbf{U}^n = \mathbf{U}^{n-1} + (\Delta t)\dot{\mathbf{U}}^{n-1} + (\Delta t^2/2)\ddot{\mathbf{U}}^{n-1} + (\gamma_2\Delta t^2)\Delta\ddot{\mathbf{U}}, \tag{7b}$$

$$\mathbf{P}^n = \mathbf{P}^{n-1} + (\Delta t)\dot{\mathbf{P}}^{n-1} + (\gamma_3\Delta t)\Delta\dot{\mathbf{P}}, \tag{7c}$$

where $\mathbf{U}^n$ stands for a numerical approximation for $\mathbf{U}(t_n)$ and $\Delta\mathbf{U}$ is defined by $\Delta\mathbf{U} = \mathbf{U}^n - \mathbf{U}^{n-1}$ (analogous definitions are considered for $\mathbf{P}^n$ and $\Delta\mathbf{P}$). For an unconditional stable scheme, the relations $\gamma_1 \geqslant 0.5$, $\gamma_2 \geqslant 0.5\gamma_1$ and $\gamma_3 \geqslant 0.5$ must hold in Eqs. (7).

By taking into account Eqs. (7) and by introducing an iterative procedure (e.g., $\Delta\mathbf{U}_{(k+1)} = \Delta\mathbf{U}_{(k)} + \Delta\,\Delta\mathbf{U}_{(k+1)}$) in order to numerically treat the nonlinear system of Eqs. (3) and (4), the following final system of equations can be obtained:

$$\begin{bmatrix} (1/(\gamma_2\Delta t^2))\mathbf{M} + \mathbf{K}_T & -\mathbf{Q} \\ -\mathbf{Q}^T & -(\gamma_2\Delta t/\gamma_1)\mathbf{H} - (\gamma_2/(\gamma_3\gamma_1))\mathbf{S} \end{bmatrix} \begin{bmatrix} \Delta\Delta\mathbf{U}_{(k+1)} \\ \Delta\Delta\mathbf{P}_{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^u_{(k)} \\ \mathbf{F}^p_{(k)} \end{bmatrix}, \tag{8}$$