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A type of high order schemes for steady convection-diffusion problems

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ABSTRACT

Based on the viewpoint that the flow direction can be accounted for by adding more weigh of the contribution of the upstream grid point in evaluating the transport fluxes at the cell faces for the convection-dominated flows, a type of high order schemes are proposed for solving convection-diffusion equations. These schemes share the same main algorithm which is developed by integrating the convection-diffusion equation expressed as the first order partial differential equations over selected regions and applying the numerical quadrature with a weighting parameter for approximating the resulting integrals. Interestingly, the error analysis allows us to determine the expressions of the weighting parameter for a series of schemes with the third order scheme as the lowest and the infinite order as the highest order scheme for the source free convection-diffusion problems. Numerical results show that the highest order scheme achieves almost the same accuracy as the exact solution, and does not induce any unphysical oscillation for the convection-dominated flows. Even the third order scheme shows obvious advantages over the traditional finite volume method (FVM) and QUICK scheme in dealing with the multi-dimensional convection-diffusion problems.

1. Introduction

Many important phenomena in fluid dynamics and magnetohydrodynamics are governed by convection-diffusion equations [1–4]. Since the exact solution of the convection-diffusion equation is only available for the simple geometries and boundary conditions and velocity fields, numerical solution is the main approach for solving the convection-diffusion problems. Various numerical methods, such as the finite difference method (FDM) [5–14], FVM [15–20] and finite element method (FEM) [21–26], have been widely applied in numerical simulation of the convection-diffusion problems. Among them, the FVM has played an important role in the most well-established CFD codes. At the second-order level the FDM and FVM are essentially equivalent. The traditional FVM based on the central differencing scheme is only suitable for the diffusion-dominated low Reynolds number flows. For the convection-dominated flows, it may induce unphysical oscillations. Although the basic upwind differencing (UD) scheme is the most stable and unconditionally bounded scheme, its low order of accuracy introduces a high level of false diffusion. In order to avoid the unphysical oscillations near discontinuities, popular schemes have been built on Godunov-type discretizations based on the Total Variation Diminishing (TVD) property [27]. The TVD solutions of convection-diffusion problems show far less false diffusion than UD scheme. Moreover, they do not introduce any unphysical overshoots and undershoots. However, all TVD schemes will degenerate to lower order accuracy near local smooth extrema [28]. In addition, the TVD scheme would consume more CPU-time in comparison with an ordinary scheme [43].

Based on a different approach, Harten et al. -proposed a self similar, uniformly high accurate, and essentially non-oscillatory (ENO) interpolation for piecewise smooth function in solving hyperbolic partial differential equations [29]. Later, the weighted ENO (WENO) schemes were developed by using a convex combination of all candidate stencils instead of just one as in the original ENO [30]. The WENO reconstructions are very successful in capturing shocks in a non-oscillatory fashion while maintaining high accuracy in smooth regions. However, Shen and Zha found that the fifth-order WENO scheme will degenerate to the third order at a transition point near discontinuities [31]. In addition, the WENO schemes require uniform or at lease smoothly varying mesh size [32]. Furthermore, the WENO schemes are not favorable to simulate the small scale turbulent flows [33]. As pointed out by Skála et al., the current high order schemes are numerically expensive, and the changes and modifications of initial and boundary conditions also require quite some effort [34]. Based on this viewpoint, Skála et al. just took the simple second-order leap-frog discretization scheme as the solver for developing a three-dimensional magnetohydrodynamical code [34]. Therefore, it is still of great value to design numerical methods which are as simple as the traditional

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FVM on the one hand, and accurate and robust on the other hand for simulating the complicated phenomena of fluid dynamics and manetohydrodynamics, such as turbulence, magnetic reconnection and dynamo actions. In order to seek some clues for achieving this goal, we start with analyzing the FVM and Galerkin finite element method from a different viewpoint.

The control volume integration which is a key step in the FVM can keep the conservation of the relevant properties for each finite size cell, which is one of main advantages of the FVM. In fact, the control volume integration has another advantage that it reduces the order of the highest derivative that appears in the governing equations of fluid flows and heat transfers, which weakens the requirement of the smoothness of the unknown function. Actually, the Galerkin finite element method [22,24,26] demonstrates the similar merit. Along this direction it is desirable to transform the governing equation of the convection-diffusion problem in the partial differential equation form to a pure integral equation form. Based on this viewpoint some integration methods for the convection-diffusion problems have already been developed. An axial Green's function method (AGM) was proposed for solving the multi-dimensional elliptic boundary value problems [35]. Later, it was extended to simulate the Stokes flow [36]. Recently, a local axial Green's function method which is the localization of the AGM was established for solving the convection-diffusion equation [37]. Similarly, a nonstandard finite difference scheme based on the Green's function formulation was proposed for solving the reaction-diffusion-convection problems [38]. Recently, a finite integration method was proposed for solving partial differential equations by using numerical quadrature or radial basis function interpolation [39]. Based on the Green's function in a series form and the integration formulation, we have designed an integral equation approach for simulating the steady hydromagnetic dynamo [40], the magnetic reconnection phenomena [41] and the convection-diffusion problems [42]. In the following, we attempt to develop a simple and efficient numerical method with a high accuracy to solve the convection-dominated transport problems.

2. Governing equation and discretization algorithm

For the sake of clarity in the description of the algorithm and without loss of generality, consider the following two-dimensional steady convection-diffusion equation:

$$\frac{\partial}{\partial x}(\rho u\phi)+\frac{\partial}{\partial y}(\rho v\phi)=\frac{\partial}{\partial x}\left(\Gamma_x\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma_y\frac{\partial\phi}{\partial y}\right)+S \tag{1}$$

where (x,y) is the position coordinate in the Cartesian coordinate system, ρ is the density, u and v are respectively the velocity components along x- and y-direction, ϕ is a conserved property, Γ_x and Γ_y are respectively the diffusion coefficients in x- and y-directions, S is the source term. Eq. (1) can be rewritten into the following form:

$$\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}=S \tag{2}$$

where J_x and J_y are the fluxes along x- and y-directions, respectively. From Eqs. (1) and (2), the fluxes can be expressed as follows:

$$J_x=\rho u\phi-\Gamma_x\frac{\partial\phi}{\partial x} \tag{3a}$$

$$J_y=\rho v\phi-\Gamma_y\frac{\partial\phi}{\partial y} \tag{3b}$$

Denote the fluid medium with an arbitrary geometry as V as shown in Fig. 1. The point (x_{i+1},y_{j+1}) is a representative node point. Its adjacent node points are (x_i,y_{j+1}), (x_{i+2},y_{j+1}), (x_{i+1},y_j), and

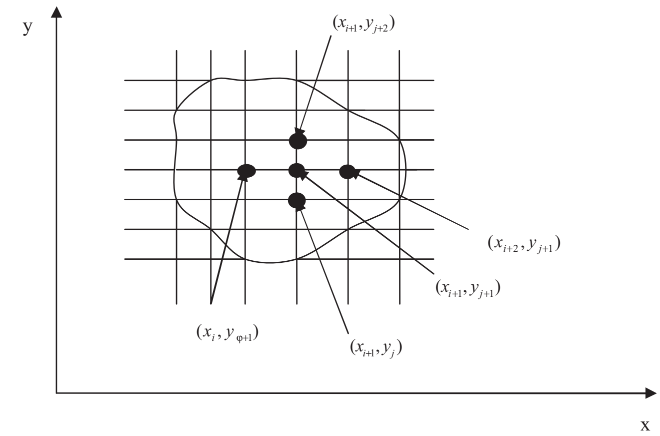

Fig. 1. Diagram of the convection-diffusion domain and grid points.

(x_{i+1},y_{j+2}) as depicted in Fig. 1. Denote $x_{i+1/2}$ and $y_{j+1/2}$ as $x_i+0.5(x_{i+1}-x_i)$ and $y_j+0.5(y_{j+1}-y_j)$, respectively. Integrating Eq. (2) over the small domain $[x_{i+1/2},x_{i+3/2}]\times[y_{j+1/2},y_{j+3/2}]$ around the node point (x_{i+1},y_{j+1}) yields

$$\int_{x_{i+1/2}}^{x_{i+3/2}}\int_{y_{j+1/2}}^{y_{j+3/2}}\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}dxdy=\int_{x_{i+1/2}}^{x_{i+3/2}}\int_{y_{j+1/2}}^{y_{j+3/2}}Sdxdy$$

By the rule of the integration by part, we obtain

$$\int_{y_{j+1/2}}^{y_{j+3/2}}J_x(x_{i+3/2},y)-J_x(x_{i+1/2},y)dy+\int_{x_{i+1/2}}^{x_{i+3/2}}J_y(x,y_{j+3/2})-J_y(x,y_{j+1/2})dx=S_{i+1,j+1} \tag{4}$$

where $S_{i+1,j+1}=\int_{x_{i+1/2}}^{x_{i+3/2}}\int_{y_{j+1/2}}^{y_{j+3/2}}Sdxdy$. Note that in Eq. (4) there are no spatial derivatives of the fluxes. It can be viewed as a pure integral equation. Application of the mid-point formula on the integrals in Eq. (4) yields

$$[J_x(x_{i+3/2},y_{j+1})-J_x(x_{i+1/2},y_{j+1})](y_{j+3/2}-y_{j+1/2})+[J_y(x_{i+1},y_{j+3/2})-J_y(x_{i+1},y_{j+1/2})](x_{i+3/2}-x_{i+1/2})=S_{i+1,j+1} \tag{5}$$

Next integrating Eq. (3a) over $[x_i,x_{i+1}]$ gives

$$\int_{x_i}^{x_{i+1}}J_xdx=\int_{x_i}^{x_{i+1}}\rho u\phi dx-\Gamma_{x,i+1}\phi_{i+1}+\Gamma_{x,i}\phi_i+\int_{x_i}^{x_{i+1}}\frac{\partial\Gamma_x}{\partial x}\phi dx \tag{6}$$

Notice that Eq. (6) is also a pure integral equation. It can be approximated in the following way:

$$\begin{aligned}J_{x,i+1/2}&=(1-\alpha_1)\rho_iu_i\phi_i+\alpha_1\rho_{i+1}u_{i+1}\phi_{i+1}\\&+\left(\Gamma_{x,i}\phi_i-\Gamma_{x,i+1}\phi_{i+1}\right)/(x_{i+1}-x_i)+(1-\alpha_2)\frac{\partial\Gamma_{x,i}}{\partial x}\phi_i\\&+\alpha_2\frac{\partial\Gamma_{x,i+1}}{\partial x}\phi_{i+1}\end{aligned} \tag{7}$$

where $\alpha_i(i=1,2)$ are weighting parameters and satisfies $0\leqslant\alpha_i\leqslant1$, the subscripts in Eq. (7) are for the variable x, for example, $J_{x,i+1/2}=J_x(x_{i+1/2},y)$ and $\phi_i=\phi(x_i,y)$. Integrating Eq. (3a) over $[x_{i+1},x_{i+2}]$ yields

$$\begin{aligned}J_{x,i+3/2}&=(1-\alpha_1)\rho_{i+1}u_{i+1}\phi_{i+1}+\alpha_1\rho_{i+2}u_{i+2}\phi_{i+2}\\&+(\Gamma_{x,i+1}\phi_{i+1}-\Gamma_{x,i+2}\phi_{i+2})/(x_{i+2}-x_{i+1})\\&+(1-\alpha_2)\frac{\partial\Gamma_{x,i+1}}{\partial x}\phi_{i+1}+\alpha_2\frac{\partial\Gamma_{x,i+2}}{\partial x}\phi_{i+2}\end{aligned} \tag{8}$$