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Inverse analysis of heat conduction problems with relatively long heat treatment

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ABSTRACT

This paper proposes a solution method for one-dimensional inverse heat conduction problems that require a relatively long time. A hybrid technique is applied to analyze laser surface heating and spray cooling on a hot surface. In the present study, the unknown temperature in half-range expansions form is assumed, and the shifting function method is used to obtain an analytic solution. The coefficients of the half-range expansions could be determined with the least-squares method in conjunction with the analytic solution and measured temperatures. Mathematical and experimental examples are given to illustrate the analyses.

1. Introduction

Inverse heat conduction problems (IHCPs) depend on heat flux and/or temperature measurements for the estimation of unknown boundaries. The applied thermal engineering methods include laser surface heating, the use of a heat exchanger, and rapid cooling and quenching. The solution for the differential heat conduction equation is obtained from a set of boundary conditions and an initial condition. The linear boundary value heat conduction problems are separated into three groups, as outlined by Özisik [1]. However, IHCPs are often mathematically ill-posed in that the solution is highly sensitive to the input data. The solution techniques for ill-posed heat transfer problems can be viewed in these textbooks [2,3].

For one of the earliest IHCP solution, Stolz [4] used Duhamel's theorem to determine unsteady heat flux. Beck [5] calculated the surface flux based on Duhamel's theorem, and his proposed methods are similar to the methods given by Stolz [4]. Many investigators have studied one-dimensional inverse heat conduction problems. Imber and Khan [6], Monde [7], and Monde et al. [8] utilized Laplace transform techniques to obtain a closed form solution. Moreover, a regular iteration algorithm has been constructed for IHCPs. Alifanov and Mikhailov [9] studied a nonlinear generalized thermal conductivity equation and obtained the nonstationary thermal flux based on the conjugate gradient method. Wang et al. [10] used the conjugate gradient method to analyze the problem of laser heat treatment on a surface, with the LSM result serving as an initial CGM guess. The finite differences method is a widely used numerical method. Beck et al. [11] linearized nonlinear differential equations and used this algorithm to solve the sensitivity problem. Chen and Lee [12] proposed a hybrid technique for the Laplace transform and finite differences in conjunction with the least-squares method and experimental temperature data to estimate spray cooling characteristics on a hot surface.

Most of the existing methods used to solve these kinds of time-dependent inverse problems are tedious. For a time-dependent boundary condition problem, Lee and Lin [13] generalized the solution method of Mindlin and Goodman [14] and developed the shifting function method to study nonuniform beams. Lee and Yan [15] extended the shifting function method to study the exact deflection of an out-of-plane curved Timoshenko beam with nonlinear boundary conditions. Recently, an integral-transform-free solution method for one-dimensional IHCPs with time-dependent boundary conditions was presented by Lee and Huang [16,17]. They assumed the unknown temperature in polynomial function form and used the shifting function method to obtain an analytic solution. The coefficients of a polynomial function could be determined using the least-squares method in conjunction with the analytic solution and measured temperature. In the study of [17], the entire time domain is divided into several sub-time intervals.

In consideration of the entire time domain, we use the half-range expansions as the unknown time-dependent temperature in this paper. A hybrid inverse scheme of the shifting function

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method in conjunction with the least-squares method and experimental temperature data is applied to analyze laser surface heating or spray cooling on a hot surface. Consequently, the temperature distribution and the heat flux with the entire time and space domains could be obtained.

2. Mathematical formulation of 1D heat conduction

A one-dimensional boundary-value problem of heat conduction in a finite region $0 \leqslant x \leqslant L$ of a cylinder/slab can be introduced to estimate an unknown surface temperature and heat flux. In Özisik's book [1], surface temperature and heat flux, which are two important quantities in the heat conduction problem, were grouped into three patterns. The general governing differential equation, boundary conditions and initial condition are expressed as

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}, \quad \text{in } 0 \leqslant x \leqslant L,\ 0 < t < t_f \tag{1}$$

$$k_1 \frac{\partial T(x,t)}{\partial x} + h_1 T(x,t) = f_1(t), \quad \text{at } x = 0,\ 0 < t < t_f \tag{2}$$

$$k_2 \frac{\partial T(x,t)}{\partial x} + h_2 T(x,t) = f_2(t), \quad \text{at } x = L,\ 0 < t < t_f \tag{3}$$

and

$$T(x,t) = T_0(x), \quad \text{in } 0 \leqslant x \leqslant L,\ t = 0, \tag{4}$$

where x and t denote the spatial-domain and time variables, respectively. T is the temperature over the entire domain. The parameter α is called thermal diffusivity, and $\alpha = k/\rho c$, where k, ρ, and c are the thermal conductivity, density, and specific heat of the material, respectively. L is the length of the cylinder/slab. T_0 is the initial temperature. k_1 and h_1 are the thermal conductivity and the heat convection coefficients at $x = 0$, respectively. k_2 and h_2 are the thermal conductivity and the heat convection coefficients at $x = L$, respectively. t_f denotes the time that laser surface heating or spray cooling is terminated. $f_1(t)$ and $f_2(t)$ are the time-dependent temperature functions at $x = 0$ and $x = L$, respectively.

3. Analytic solutions

The general analytic solution for the differential Eq. (1) of 1D heat conduction with boundary and initial conditions (2)–(4) can be derived.

3.1. Change of variable

The shifting function method developed by Lee and Lin [13] is given below as:

$$T(x,t) = v(x,t) + \sum_{i=1}^{2} f_i(t) g_i(x), \tag{5}$$

where $g_i(x)$, $i = 1, 2$, are the shifting functions to be specified, and $v(x,t)$ is the transformed function.

Substituting Eq. (5) into Eqs. (1)–(4) yields the differential equation for $v(x,t)$ and the associated boundary conditions:

$$\frac{\partial^2 v(x,t)}{\partial x^2} - \frac{1}{\alpha}\frac{\partial v(x,t)}{\partial t} = \sum_{i=1}^{2}\left[\frac{1}{\alpha}\frac{df_i(t)}{dt} g_i(x) - f_i(t)\frac{d^2 g_i(x)}{dx^2}\right], \tag{6}$$

at $x = 0$,

$$k_1 \frac{\partial v(x,t)}{\partial x} + h_1 v(x,t) = f_1(t) - \sum_{i=1}^{2} f_i(t)\left[k_1 \frac{dg_i(x)}{dx} + h_1 g_i(x)\right], \tag{7}$$

at $x = L$,

$$k_2 \frac{\partial v(x,t)}{\partial x} + h_2 v(x,t) = f_2(t) - \sum_{i=1}^{2} f_i(t)\left[k_2 \frac{dg_i(x)}{dx} + h_2 g_i(x)\right]. \tag{8}$$

The associated initial condition is

$$v(x,0) = T(x,0) - \sum_{i=1}^{2} f_i(0) g_i(x). \tag{9}$$

3.2. Shifting functions

The shifting functions $g_i(x)$, $i = 1, 2$, in Eqs. (6)–(8) are chosen to satisfy the differential equation

$$\frac{d^2 g_i(x)}{dx^2} = 0, \tag{10}$$

and the following boundary conditions,

at $x = 0$,

$$k_1 \frac{dg_i(x)}{dx} + h_1 g_i(x) = \delta_{ij}, \quad j = 1, \tag{11}$$

at $x = L$,

$$k_2 \frac{dg_i(x)}{dx} + h_2 g_i(x) = \delta_{ij}, \quad j = 2, \tag{12}$$

where δ_{ij} is a Kronecker symbol.

These two shifting functions $g_1(x)$ and $g_2(x)$ can be easily determined as

$$g_1(x) = \frac{h_2}{k_1 h_2 - (k_2 + h_2 L) h_1} x - \frac{k_2 + h_2 L}{k_1 h_2 - (k_2 + h_2 L) h_1}, \tag{13}$$

$$g_2(x) = -\frac{h_1}{k_1 h_2 - (k_2 + h_2 L) h_1} x + \frac{k_1}{k_1 h_2 - (k_2 + h_2 L) h_1}. \tag{14}$$

With Eqs. (10)–(12), the transformed governing differential Eq. (6) of 1D heat conduction becomes

$$\frac{\partial^2 v(x,t)}{\partial x^2} - \frac{1}{\alpha}\frac{\partial v(x,t)}{\partial t} = F(x,t), \tag{15}$$

$$\text{where} \quad F(x,t) = \sum_{i=1}^{2}\left[\frac{1}{\alpha}\frac{df_i(t)}{dt} g_i(x)\right], \tag{16}$$

and the associated boundary conditions (7) and (8) can be reduced to homogeneous conditions as

$$k_1 \frac{\partial v(0,t)}{\partial x} + h_1 v(0,t) = 0, \tag{17}$$

and

$$k_2 \frac{\partial v(L,t)}{\partial x} + h_2 v(L,t) = 0. \tag{18}$$

The transformed initial condition is

$$v(x,0) = T(x,0) - \sum_{i=1}^{2} f_i(0) g_i(x) = v_0(x). \tag{19}$$

3.3. Solution of transformed variable

To use the eigenfunction expansion method, the transformed variable $v(x,t)$ in Eqs. (15)–(19) can be expressed as

$$v(x,t) = \sum_{n=1}^{\infty} \phi_n(x) B_n(t). \tag{20}$$