# On a solution strategy for energy-based mesh optimization in finite hyperelastostatics

Michael Scherer, Ralf Denzer, Paul Steinmann [*]

*Department of Mechanical Engineering, Chair of Applied Mechanics, University of Kaiserslautern, P.O. Box 3049, D-67653 Kaiserslautern, Germany*

## Abstract

The objective of this work is the development of a numerical solution strategy for energy-based mesh optimization in finite hyperelastostatics. In finite element computations that rely on the principle of minimum potential energy, the variational principle itself provides the basis for r-adaptive methods. The numerical solution can be improved by further minimizing the discrete potential energy with respect to the material node point positions. In this paper, we regard the mesh optimization as a nonlinear minimization problem with equality and inequality constraints. The equality constraints correspond to the spatial equilibrium condition, whereas the inequality constraints are given by the natural restriction that material elements with a negative volume (Jacobian) are inadmissible. Based on this interpretation, we develop a stable numerical solution strategy in which two approaches of nonlinear programming are combined. Applying a barrier method, the minimization problem is transformed into a sequence of problems without inequality constraints. Each problem of the sequence is solved by means of a Newton scheme that operates on the constrained surface given by the spatial equilibrium condition.
© 2007 Elsevier B.V. All rights reserved.

*Keywords:* r-Adaptivity; Hyperelasticity; Energy minimization; Barrier method; Configurational forces

## 1. Introduction

The numerical solution of a boundary-value problem in solid mechanics usually relies on a variational representation of the problem. Throughout this paper, we restrict ourselves to hyperelastic bodies subjected to conservative surface traction and volume forces. For such problems, the principle of minimum potential energy represents a suitable variational formulation. The variational problem and the associated boundary-value problem can be solved numerically by means of a Ritz method. A Ritz method in which a finite element approximation of the unknown deformation is used has the advantage that the underlying variational principle provides the basis for r-adaptive schemes which are within this work embraced by the term "energy-based mesh optimization". Without r-adaptivity, the discrete

potential energy is solely minimized with respect to the spatial node point positions by numerically solving the spatial equilibrium equations. As a result, one obtains a numerical solution of the boundary-value problem. To attain a more accurate solution, an energy-based mesh optimization scheme further minimizes the discrete potential energy with respect to the material node point positions.

This paper deals with a pure r-adaptive scheme that is derived from the definition of the mesh optimization as a problem of nonlinear programming. We regard the discrete potential energy as a function of the material and the spatial node point positions which has to be minimized with respect to both sets of variables. The problem is subjected to the spatial equilibrium condition, a set of equality constraints, and inequality constraints corresponding to the restriction that material elements with a negative volume (Jacobian) are inadmissible. Our solution strategy for the constrained problem combines two approaches of nonlinear programming. Applying a barrier method, the problem is transformed into a sequence of minimization problems

---

[*] Corresponding author. Tel.: +49 631 205 2419; fax: +49 631 205 2128.
*E-mail addresses:* mscherer@rhrk.uni-kl.de (M. Scherer), denzer@rhrk.uni-kl.de (R. Denzer), ps@rhrk.uni-kl.de (P. Steinmann).

without inequality constraints. To solve each problem of the resulting sequence, we use a staggered scheme, a Newton method that operates on the constrained surface given by the spatial equilibrium condition. The proposed algorithm is based on the assumption that the spatial node point positions are an implicit function of the material node point positions. "Staggered" means: The method alternately solves the equilibrium condition and shifts the material nodes.

Earlier publications dealing with linear elasticity already present several solution strategies. Note that although the system of equations for the computation of the nodal displacements is linear, the minimization of the potential energy with respect to the nodal positions is a nonlinear problem. Carroll [1], McNeice and Marcal [2], Felippa [3,4], Bathe and Sussman [5,6] use staggered schemes, steepest descent, conjugate gradient (cg) or derivative free methods. Carpenter and Zendegui [7] propose a cg method that determines the nodal positions and displacements simultaneously. Concerning the mesh distortion, the algorithms used in [4,7] also include measures that prevent the nodal positions from leaving the feasible region. A precise formulation of the underlying problem of nonlinear programming is given in [3], and an explicit expression for the gradient of the discrete potential energy with respect to the nodal coordinates is specified in [5,6].

The relation between configurational mechanics and energy-based mesh optimization for finite elasticity was first pointed out by Braun [8]. He shows that the gradient of the discrete potential energy with respect to the material node point positions can be expressed in terms of the Eshelby stress tensor. Therefore, he denotes the components of the gradient associated with the nodes as discrete configurational forces. Mueller et al. [9–11] investigated the application of configurational node point forces in the context of r,h-adaptivity, fracture mechanics and inhomogeneities. As a solution strategy for the r-adaptivity, they use a staggered steepest descent method, i.e. the material node points are shifted in the opposite direction of the configurational node point forces.

In the context of finite elasticity, energy-based mesh optimization was further investigated by Thoutireddy and Ortiz [12,13]. They use a staggered cg method and incorporate connectivity changes of the mesh to improve the performance of their scheme. The work of Thoutireddy was pursued by Mosler and Ortiz [14]. Focussing on the numerical implementation, they propose a simultaneous solution strategy based on the viscous regularization of the configurational forces and develop strategies for optimizing the mesh connectivity and allowing node migration in and out of the boundary. In the two-part publication of Kuhl, Askes and Steinmann [15,16], the considered r-adaptivity is embedded into a variational ALE formulation for finite elasticity. Therein, the variational formulation is derived from the principle of stationary potential energy. The finite element discretization of the variational formulation renders two coupled systems of equations which correspond to the derivatives of the discrete potential energy with respect to the spatial and material node point positions. To solve the resulting equations, the authors propose staggered and simultaneous Newton schemes.

This paper is structured as follows. In Section 2, we briefly reiterate the essential kinematics of nonlinear continuum mechanics. Section 3 outlines the problem and the variational principle for its numerical analysis. The construction of a finite element approximation for the spatial deformation and its usage in the framework of a Ritz method is the main topic of Section 4. Section 5 is concerned with the concept of energy-based r-adaptivity. Thereby, we focus on the definition of the underlying problem of nonlinear programming, a minimization problem subjected to equality and inequality constraints. The clear definition of the problem narrows down the search for a suitable numerical procedure. Section 6 is divided into 3 subsections: First, two approaches of nonlinear programming, one for each of the two constraints, are discussed. Then, we illustrate the resulting algorithm, a combination of the selected schemes. In Section 7, two numerical examples, a cracked specimen and a plate with hole, show the performance of our solution strategy. A discussion in Section 8 completes the paper.

## 2. Kinematics

In order to introduce terminology and notation, we briefly reiterate some key issues concerning kinematics, balance laws, the variational formulation and the discretization. For detailed information about the named topics we refer to [17–20]. In nonlinear elastostatics the time-independent spatial motion of a body is described by the deformation map

$$\boldsymbol{\varphi} : \mathscr{B}_0 \to \mathscr{B}_t, \quad \text{i.e. } \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}), \tag{1}$$

see Fig. 1. The function $\boldsymbol{\varphi}(\boldsymbol{X})$ maps each point $\boldsymbol{X}$ of the stress free material configuration $\mathscr{B}_0 \subset \mathbb{E}^3$ to a point $\boldsymbol{x}$ of the spatial configuration $\mathscr{B}_t \subset \mathbb{E}^3$, where $\mathbb{E}^3$ denotes the three dimensional Euclidean space. Since the spatial deformation map $\boldsymbol{\varphi}(\boldsymbol{X})$ is assumed to be one-to-one, an inverse mapping exists

$$\boldsymbol{\Phi}(\boldsymbol{x}) : \mathscr{B}_t \to \mathscr{B}_0, \quad \text{i.e. } \boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x}), \tag{2}$$
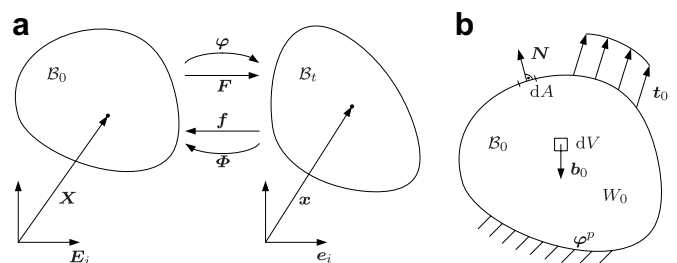


Fig. 1. Kinematics and boundary-value problem.