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# A random field model based on nodal integration domain for stochastic analysis of heat transfer problems

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In this paper, a random field model based on nodal integration domain is presented to solve stochastic heat transfer problems. In the proposed model, the uncertainty of the inputs are considered as random field, which is discretized into a number of node-based subdomains and the properties of the uncertainties under random field can be considered at the nodes. The proposed method is efficient to model non-uniform material under random field with constitutive equation, meanwhile, the random field with arbitrary geometry can be simulated conveniently and efficiently by using the Karhunen-Loève expansion truncated in this work. The statistical moments of the structural responses using the perturbation method is also performed and compared with the solutions of Monte Carlo simulation. The proposed method is successfully applied to the steady-state heat transfer problem with spatially varying random material parameter introduced in the thermal conductivity in this work. Finally, we demonstrate the accuracy and performance of the proposed method through a series of numerical examples both in 2D and 3D steady-state heat transfer problems under different random fields.



## 1. Introduction

Thermal analyses have made great progress in many works [1–10] but most under the assumption that the properties and boundary conditions are deterministic. Therefore, it's limited to describe the general characteristics of a system. In particular, it cannot directly study a system reliability where there exists some degree of uncertainty. In reality the parameters can be truly random due to heterogeneity of the material or variations in time or a reflection of our uncertain knowledge about their solicitations, such as potential thermal gradients or pressure loads. Therefore, it is reasonable to treat materials, boundary conditions, geometry sizes and external loads as stochastic parameters and formulate the uncertain analysis, especially in the precise reliable problems.

Previous models of stochastic heat transfer problems have been discussed in which the thermal conductivity is spatially random [11,12], the internal heat generation has spatial or temporal randomness [13,14], the boundary conditions or the initial conditions vary randomly [15,16] and the shape and material properties [17,18] are random, etc. In recent years, the concept of stochastic simulation techniques are being used for the analysis of heat transfer and thermoelastic problem due to the fact that stochastic thermal analysis is more necessary to maintain the reliability and safety gain in the design of high-temperature devices or heat resistant structures. Therefore, it appears to be more realistic. Stochastic finite element method [19], which combines the classic FEM and other methodologies, has been developed and grown in importance over time. Monte Carlo simulation (MCs) is the simplest and versatile probabilistic method in the framework of stochastic methods which requires the most computational power. Even so, MCs is widely accepted and is often used to validate the perturbation method and the spectral stochastic finite element method (SSFEM) [20]. The perturbation method is another popular branch of the stochastic finite element method which uses Taylor series expansions to introduce randomness into the system and estimates the influence of the mean, standard deviation and covariance of response variables of a structure [21,22]. The spectral stochastic finite element method [23,24] which uses the Karhunen-Loève expansion or polynomial chaos expansions is mainly concerned with representing the random material properties of a structure. The SSFEM can provide an efficient means of calculating statistical moments (e.g. mean and variance) and probability distributions

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associated with the probabilistic behavior of a structure, and its computational efficiency is in some cases greater than that of MCs, therefore it would be suitable as an alternative to Monte Carlo simulation for assessing structural reliability.

The SSFEM have received more attention recently, Since its early inception, further developments in efficient algorithms, capabilities and performances of the original SSFEM have been developed and extended for stochastic problems in recent years [25–27]. These studies have inspired the improvement of a reliable computational procedure in terms of uncertain modeling for different parameters in SSFEM. Indeed, most of the stochastic finite element methods are based on the traditional FEM. However, the FEM always provide a poor accuracy when low-order linear element is used and when the element mesh is heavily distorted. In order to cure these defects, many modified numerical algorithms based on the finite element framework have been developed and have made great progress [28–37]. A class of smoothed FEM models have been developed on the base of the gradient smoothing technique, such as the smoothed finite element method [38–41], the node-based smoothed finite element method (NS-FEM) [42–47], the edge-based smoothed finite element method (ES-FEM) [48–51], the face-based smoothed finite element method (FS-FEM) [52], and the stable node-based smoothed finite element method (SNS-FEM) [53–55].

Suitable random model corresponding to real stochastic engineering problems is still an open problem, meanwhile, the efficiency in modeling uncertainties is also a profound topic in solving stochastic heat transfer problem. When there is a relatively strong correlation between the material properties in two adjacent points, it is desirable to consider the material property of each discrete subdomain as a random variable that can be defined as the value of the KL expansion at the center $\bar{x}$ of this subdomain, fortunately, the centroid $\bar{x}$ of the subdomain $\Omega_k^s$ is just at the node $k$ in the NS-FEM. It is more convenient than some other similar numerical schemes in terms of FEM which need to get the centroid of each subdomain, for example, the scaled boundary finite element method combined with the random field theory [56]. Hence, it provides an effective way to combine the NS-FEM with the random field theory. In this paper, a random field model based on nodal integration domain is proposed for stochastic analysis of heat transfer problems. This random field is constructed by discreting the analyzed random domain into a number of node-based subdomains. Using gradient smoothing technique over the cells associated with nodes (i.e., NS-FEM), line integrations along the edges of the cells is performed and no limitation of elements used herein, then the properties of the nonuniform material can be considered at the nodes which is more efficient and simpler when representing the random field and establishing the constitutive equation. The proposed method is efficient to model non-uniform material under random field with constitutive equation, meanwhile, the random field with arbitrary geometry can be simulated conveniently and efficiently by using the Karhunen-Loève expansion truncated in this work. The statistical moments of the structural responses using the perturbation method is also performed and compared with the solution of Monte Carlo simulation in this work. So the present method has the large potential to apply into reliability analysis.

In this paper, the proposed method is applied to deal with the stochastic steady-state heat transfer problems with random thermal conductivity. A schematic of the general procedure of the proposed method is depicted in Fig. 1. The remainder of this paper is organized as follows. Section 2 describes the basic theory of the steady-state heat transfer problems and the discretization of random field using NS-FEM; Section 3 shows the representation of the random field using Karhunen-Loève expansion; Sections 4 represents the perturbation method for statistical analyses; Section 5 provides the analyses of four numerical examples.

## 2. Discretized system equations

### 2.1. Standard Galerkin weak form

In this work, the gradient smoothing technique is formulated for steady-state heat transfer problems without considering thermal-mechanical coupling and inertia. The governing equation and boundary conditions represented by the tensor notation are [57]:

$$(k_{ij}T_{,j})_{,i} + Q_v = 0 \quad \text{Problem domain studied} \tag{1}$$

$$T = T_0 \quad \text{Initial condition} \tag{2}$$

$$T = T_\Gamma \quad \text{Dirichlet boundary} \tag{3}$$

$$-n_i k_{ij} T_{,j} = q \quad \text{Neumann boundary} \tag{4}$$

$$-n_i k_{ij} T_{,j} = h(T - T_a) \quad \text{Robin boundary} \tag{5}$$

$$n_i k_{ij} T_{,j} = 0 \quad \text{Adiabatic boundary} \tag{6}$$

where $k_{ij}$ is the thermal conductivity, $Q_v$ is internal heat source, $T_0$ is the initial temperature of the domai$_\text{n}$, $T_\Gamma$ is the temperature on the boundary of domain, $n_i$ is component of the unit outward normal to the boundary, $q$ is the prescribed heat flux, $h$ is the convective heat transfer coefficient and $T_a$ is the temperature of surrounding medium.

The standard Galerkin weak form can be writted as:

$$\begin{aligned}&\int_\Omega (\mathbf{L}\delta T)^T \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} (\mathbf{L}T) d\Omega - \int_\Omega \delta T^T Q d\Omega + \int_{\Gamma_2} \delta T^T q d\Gamma \\ &+ \int_{\Gamma_3} \delta T^T h (T - T_a) d\Gamma \\ &= 0\end{aligned} \tag{7}$$

where $\mathbf{L}$ is a differential operator in the following form:

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}^T \tag{8}$$

In the above Galerkin weak form Eq. (7), the temperature field $T$ can be expressed in an approximate form:

$$T = \sum_{i=1}^{m} \mathbf{N}_i T_i \tag{9}$$

where $T_i$ is the temperature at the node $i$, $\mathbf{N}_i$ is the shape function. Substituting Eq. (9) into Eq. (7), we can obtain the following expression:

$$\begin{aligned}&\int_\Omega \left[ \left(\frac{\partial \mathbf{N}}{\partial x}\right)^T k_x \frac{\partial \mathbf{N}}{\partial x} + \left(\frac{\partial \mathbf{N}}{\partial y}\right)^T k_y \frac{\partial \mathbf{N}}{\partial y} + \left(\frac{\partial \mathbf{N}}{\partial z}\right)^T k_z \frac{\partial \mathbf{N}}{\partial z} \right] \mathbf{T} d\Omega + \int_{\Gamma_2} \mathbf{N}^T q d\Gamma \\ &+ \int_{\Gamma_3} \mathbf{N}^T \mathbf{N} \mathbf{T} d\Gamma - h \int_{\Gamma_3} \mathbf{N}^T \mathbf{T}_a d\Gamma - \int_\Omega \mathbf{N}^T Q d\Omega = 0\end{aligned} \tag{10}$$

The discretized system equilibrium equation can be finally