Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 197 (2008) 1204–1218
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Continuum modeling of supply chain networks using discontinuous Galerkin methods

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Received 28 August 2007; accepted 23 October 2007
Available online 4 November 2007

Abstract

Using a connectivity matrix, we establish a continuum modeling approach with partial differential equations of conservation laws for simulating materials flow in supply chain networks. A number of existing and new constitutive relationships for modeling velocity are summarized or proposed. To effectively treat strong advection components within the modeling system, we apply discontinuous Galerkin (DG) methods for solving production flow in a supply chain network. In addition, a number of DG properties are analyzed for treating network flow. In particular, a nearly optimal error estimate is obtained using a new estimating technique that utilizes two physical meaningful assumptions on the connectivity matrix. Numerical examples are provided to simulate a single node, a serial supply chain and an entire network as well as to investigate the influence of influx variation and node shut-down to the profiles of work in progress (WIP) and outflux. It is shown that the proposed modeling approach is applicable to a large number of scenarios including re-entrant lines and the proposed DG algorithm is robust and accurate for predicting WIP and outflux behaviors.

Keywords: Supply chain network; Re-entrant line; Connectivity matrix; Discontinuous Galerkin method; Conservation law; Continuum modeling

1. Introduction

A supply chain can be viewed as a network of suppliers, manufacturing sites, distribution centers, and customer locations, through which components and products flow. A node in a network can be a physical location, a sub-network, or just an operation process, while links represent materials (components or products) flow. These networks find significant applications in manufacturing and logistics in many fields, such as electronic and automobile industries [15]. A central problem in integrated supply chain network design is to model and evaluate the performance of supply chains. The problem becomes more challenging because of the dynamic nature of the supply chains: prolific product variety, short lifetime products, frequent new product introduction, non-stationary customer demand, and frequently changing service-level requirements. This dynamic nature of complex supply chains causes the models change over time. In turn, the performance of supply chains must be continually reevaluated.

Much progress has been made to characterize the dynamics of supply chains. Discrete event and agent based models are routinely developed to study the dynamics of flows through such networks [13,14]. However, throughout these networks, there are different sources of uncertainties, including supply (availability and quality), process (machine break-down, operator variation), and demand (arrival time and volume). Moreover, these variations will propagate from upstream to downstream stages. These uncertainties degrade the performances of a network causing, for example, longer cycle times and lower fill-rates. Their effective allocation and control impose a great challenge to the managers of supply chains. Performance modeling and analysis become increasingly more important but difficult in the management of such complex supply chains.

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0045-7825/$ - see front matter
doi:10.1016/j.cma.2007.10.012

The traditional modeling and analysis methods such as discrete event simulations are prohibitively expensive to maintain and are not equipped well to answer questions on the behavior of the networks as a whole. In particular, discrete event simulators have increasing numerical complexity as the number of simulated parts increases, leading to computationally difficult or even intractable tasks for simulating high-volume, multistage production flow. Therefore, a continuum modeling approach to be used for modeling and performance analysis is in need. Furthermore, to answer "what-if" questions quickly, such a continuum model has to be computed efficiently.

Classical continuous models for supply chains (see e.g., [2]) use rate equations to describe the queueing and flow process in the system macroscopically. These continuous approaches can be solved efficiently by numerical computation, even though they are less exact than discrete event simulations. They are computational scalable with respect to the number of parts; in other words, their computational complexity does not depend on the number of parts to be processed. Borrowing techniques from gas dynamics, a continuum modeling approach was established in [3,5] with compromise made between rate equations and discrete event simulations. This allows scalable and efficient computation while being capable of providing more information than simple rate equations. Similarly, continuous production flow through a re-entrant factory was modeled using a continuum model in [4], where a conservation law was developed for a continuous density variable and a state equation was assumed for the speed of the flow along the production line, allowing fast and accurate simulations. Existence of solutions for continuous models on a network has also been analyzed for simple scenarios [17]. Continuum models can be also combined with discrete models [20] to take the advantage of both approaches, namely the accuracy offered by discrete models and the scalable complexity of computation offered by continuum models. In addition to their efficient computation, continuum models can be treated with a rich collection of mathematical tools available for differential equations. For example, a direct application of multiscale algorithm to the solution of continuum models can be used to build up a multiscale analysis of supply chains [33]. Here, unlike discrete models, the modeling of supply chain networks with continuum approaches does not explicitly incorporate the number of parts into the equation system. Consequently, continuum models or combination of continuum and discrete models are more suitable for multiscale modeling and cross-scale computation than the most microscopic discrete simulations. As supply chain networks naturally exhibit multiscale behaviors, easy extendability to multiscale modeling and simulation is obviously an attractive feature.

To solve continuum models, a differential equation solver is in need. As materials flow in supply chains consists of mainly advection processes and discontinuous Galerkin (DG) methods have superior numerical performance for advection-dominated problems, we utilize DG in this paper. DG methods [6–10,16,18,19,21–24,26,27,29–32] are specialized finite element methods that utilize discontinuous piecewise polynomial spaces to approximate the solutions of differential equations, with inter-element continuities (if diffusion presents) and boundary conditions weakly imposed through bilinear forms. Derived from variational principles by integration over local cells, the methods are locally mass conservative by construction. Weak enforcement of boundary conditions and inter-element continuities leads to small numerical diffusion and little oscillation for DG. In addition, the DG methods handle rough coefficient problems and capture the discontinuity in the solution very well by the nature of discontinuous function spaces. For time-dependent problems in particular, their mass matrices are block diagonal, providing a substantial computational advantage if explicit time integrations are used. Examples of DG methods include local discontinuous Galerkin [10,11], Symmetric Interior Penalty Galerkin (SIPG) [25,28,32], Oden–Babuška–Baumann DG formulation (OBB-DG) [21], Non-symmetric Interior Penalty Galerkin (NIPG) [23] and Incomplete Interior Penalty Galerkin (IIPG) methods [12,25,28]. For advection–reaction problems without diffusion, the above five DG schemes coincide.

Random variation exists in all production systems, due to various sources of uncertainties from supply, process and demand, and it can significantly affect the performance of supply chains. Incorporation of random processes into continuum models will result in stochastic partial differential equation (PDE) systems. In most numerical treatments for stochastic PDEs, including the implementation-friendly Monte Carlo method, and the more efficient Karhunen–Loeve expansion method, the stochastic PDE problem is approached by solving a number of associated deterministic PDEs. Consequently, deterministic PDE modeling and algorithms are of interest even for intrinsically stochastic systems. In this paper, we restrict our attention to deterministic PDE modeling of supply chains and its efficient solutions. Extension of this study to stochastic PDE modeling is currently under progress, and it will be presented elsewhere.

The paper is organized as follows: In the following section, we formulate the continuum modeling equations for supply chain networks. Here we first establish the mass conservation equation to model a single node in a supply chain network and formulate various constitutive equations for the velocity. We then extend this framework to a serial supply chain and to an entire supply network with a new tool of connectivity matrices. In Section 3, we propose discontinuous Galerkin methods for the numerical treatment of the modeling system and analyze various algorithmic properties of the proposed algorithm for solving an entire supply chain network. These algorithmic properties include consistency, existence of a solution, element-wise conservation, and convergence. To our knowledge, the error analysis of DG here for a PDE system equipped with a connectivity matrix is new. Section 4 is devoted to