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Unphysical effects of the dual-phase-lag model of heat conduction: higher-order approximations

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Two models of heat conduction, based on higher-order approximations to the dual-phase-lag constitutive relation, are considered. Initial value problems for equations, describing the models, are studied. Stable solutions to the problems manifest clear unphysical behavior with negative values of temperature. This implies that the equations cannot serve as appropriate models of heat conduction.

1. Introduction

The dual-phase-lag (DPL) model of heat conduction was proposed in Refs. [1,2] as an improved theory compared to the classic model, based on Fourier's law and described by the heat conduction equation. Fourier's law is valid under the assumption of local thermodynamic equilibrium, which fails in very small dimensions and short time scales [3] as well as the classic model. The cornerstone of the DPL model is the constitutive relation

$$\boldsymbol{q}(\boldsymbol{x}, t+\tau_q) = -k\nabla T(\boldsymbol{x}, t+\tau_T), \tag{1.1}$$

where τ_q and τ_T are positive time (or phase) lags. If $\tau_q = 0$ and $\tau_T = 0$ Eq. (1.1) becomes Fourier's law.

Eq. (1.1) is equivalent to the single-phase-lag (SPL) constitutive relation

$$\boldsymbol{q}(\boldsymbol{x}, t+\tau) = -k\nabla T(\boldsymbol{x}, t). \tag{1.2}$$

with $\tau = \tau_q - \tau_T$. Both relations are sensible only if $\tau \geq 0$. Initial value problems in the framework of the SPL model are ill-posed (with unstable solutions) [4–6]. Therefore, the phase-lag constitutive relations (1.1) and (1.2) *per se* have no real physical meaning.

A first-order approximation to Eq. (1.1) with respect to both τ_q and τ_T is given by the Jeffreys-type constitutive relation [7,8]

$$(\tau_q\partial_t + 1)\boldsymbol{q} = -k(\tau_T\partial_t + 1)\nabla T. \tag{1.3}$$

This relation and the energy equation

$$\partial_t T + \mathrm{div}\boldsymbol{q} = Q, \tag{1.4}$$

where $Q \equiv Q(\boldsymbol{x}, t)$ is the heat source term, and the volumetric heat capacity is equated, for simplicity, to unity, yield the Jeffreys-type equation [3,7,9]

$$\left(\tau_q\partial_t^2 + \partial_t\right)T - k(\tau_T\partial_t + 1)\Delta T = (1 + \tau_q\partial_t)Q. \tag{1.5}$$

The DPL model of heat conduction in the form of the Jeffreys-type equation received widespread attention, see Ref. [10] and references therein. However, physical anomalies and unphysical effects in the framework of this model were reported in Refs. [10–12]. Therefore, this model cannot in general serve as an appropriate model of heat conduction.

Besides the first-order approximation (1.3) higher-order approximations, leading to higher-order DPL models, were also considered in literature, see Refs. [5,13–21]. A second-order approximation in τ_q and a first-order approximation in τ_T yield the constitutive relation

$$\left(\frac{1}{2}\tau_q^2\partial_t^2 + \tau_q\partial_t + 1\right)\boldsymbol{q} = -k(\tau_T\partial_t + 1)\nabla T.$$

This relation and the energy Eq. (1.4) yield the equation

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$$\left(\frac{1}{2}\tau_q^2\partial_t^3+\tau_q\partial_t^2+\partial_t\right)T-k(\tau_T\partial_t+1)\Delta T=\left(1+\tau_q\partial_t+\frac{1}{2}\tau_q^2\partial_t^2\right)Q. \tag{1.6}$$

In Refs. [13,14] it was found that solutions of this equation are stable if $\tau_q<2\tau_T$ and unstable if $\tau_q>2\tau_T$. This means that in the latter case the solutions do not, in general, exist, and, therefore, Eq. (1.6) cannot serve as a physical model.

Second-order approximations both in τ_q and τ_T yield the constitutive relation

$$\left(\frac{1}{2}\tau_q^2\partial_t^2+\tau_q\partial_t+1\right)\boldsymbol{q}=-k\left(\frac{1}{2}\tau_T^2\partial_t^2+\tau_T\partial_t+1\right)\nabla T.$$

This relation and the energy Eq. (1.4) yield the equation

$$\left(\frac{1}{2}\tau_q^2\partial_t^3+\tau_q\partial_t^2+\partial_t\right)T-k\left(\frac{1}{2}\tau_T^2\partial_t^2+\tau_T\partial_t+1\right)\Delta T=\left(1+\tau_q\partial_t+\frac{1}{2}\tau_q^2\partial_t^2\right)Q. \tag{1.7}$$

In Ref. [14] it was found that solutions of this equation are stable if $\tau_q<(2+\sqrt{3})\tau_T$ and unstable if $\tau_q>(2+\sqrt{3})\tau_T$. This means that in the latter case the solutions do not, in general, exist, and, therefore, Eq. (1.7) cannot serve as a physical model. In Ref. [20] more restrictive thermodynamic conditions $(2-\sqrt{3})\tau_T<\tau_q<(2+\sqrt{3})\tau_T$, following from the second law, were established.

To the best of the author's knowledge there is still no study devoted to the question whether the higher-order approximations to the DPL model preserve nonnegative temperature?

In this paper an answer to this question is provided. We consider initial value problems for Eqs. (1.6) and (1.7) are considered. These problems differ from the problem studied in Ref. [10]: the fundamental difference is in the equations. The stable solutions to both the problems manifest clear unphysical behavior with negative values of temperature.

2. Initial value problem for Eq. (1.7) in 3D with a short positive localized source

Consider Eq. (1.7) in the three-dimensional space

$$\left(\frac{1}{2}\tau_q^2\partial_t^3+\tau_q\partial_t^2+\partial_t\right)T-k\left(\frac{1}{2}\tau_T^2\partial_t^2+\tau_T\partial_t+1\right)\Delta T=\left(1+\tau_q\partial_t+\frac{1}{2}\tau_q^2\partial_t^2\right)Q,\quad \boldsymbol{x}\in\mathbb{R}^3,\quad t>-\theta, \tag{2.1}$$

with a positive Gaussian source of finite duration

$$Q(\boldsymbol{x},t)=p_\sigma(\boldsymbol{x})\chi_{[-\theta,0]}(t), \tag{2.2}$$

where

$$p_\sigma(\boldsymbol{x})=\frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^3}\exp\left(-\frac{|\boldsymbol{x}|^2}{2\sigma^2}\right)$$

and

$$\chi_{[-\theta,0]}(t)=\begin{cases}\dfrac{1}{\theta}, & -\theta<t\le 0,\\ 0, & t>0.\end{cases}$$

Note that the initial time is $t=-\theta$, the source is equal to zero for $t>0$ and normalized to unity. Initial conditions for Eq. (2.1) are

$$T\big|_{t=-\theta}=0,\quad \partial_t T\big|_{t=-\theta}=0,\quad \partial_t^2 T\big|_{t=-\theta}=0. \tag{2.3}$$

The solution to the problem (2.1) and (2.3) is given by

$$T(\boldsymbol{x},t)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}T(\xi,t)e^{-i\xi\boldsymbol{x}}d\xi,$$

where

$$\mathcal{F}T(\xi,\cdot)=\int_{\mathbb{R}^3}T(\boldsymbol{x})e^{i\xi\boldsymbol{x}}d\boldsymbol{x}$$

is the Fourier transform of the solution, which is given by (see Appendix A)

$$\mathcal{F}T(\xi,t)=\frac{e^{-\sigma^2|\xi|^2/2}}{\theta}\times\int_t^{t+\theta}\left\{Ee^{\lambda_1 t'}+e^{\lambda_2 t'}\left[F\cos Bt'+\frac{G}{B}\sin Bt'\right]\right\}dt'\equiv e^{-\sigma^2|\xi|^2/2}\times\frac{1}{\theta}\left\{E\frac{e^{\lambda_1 t'}}{\lambda_1}+\frac{e^{\lambda_2 t'}}{\lambda_2^2+B^2}\left[F(\lambda_2\cos Bt'+B\sin Bt')+\frac{G}{B}(\lambda_2\sin Bt'-B\cos Bt')\right]\right\}\Big|_{t'=t}^{t+\theta},\quad t>0, \tag{2.4}$$

$\lambda_{1,2}$ are given by Eq. (A.13), and other coefficients are calculated by Eqs. (A.7)–(A.12). In the limit $\theta\to 0$ the Fourier transform is given by

$$\mathcal{F}T(\xi,t)=e^{-\sigma^2|\xi|^2/2}\left[Ee^{\lambda_1 t}+e^{\lambda_2 t}\left(F\cos Bt+\frac{G}{B}\sin Bt\right)\right],\quad t>0. \tag{2.5}$$

Due to the spherical symmetry the solution can be recast as

$$T(\boldsymbol{x},t)\equiv T(r,t)=\frac{1}{2\pi^2 r}\int_0^\infty \rho\,\mathcal{F}T(\rho,t)\sin(r\rho)d\rho \tag{2.6}$$

with

$$\mathcal{F}T(\rho,t)\equiv\mathcal{F}T(\xi,t),\ r=|\boldsymbol{x}|,\ \rho=|\xi|.$$

Figs. 1 and 2 present the solutions to the problem (2.1) and (2.3) obtained with the dimensionless parameters $k=1$, $\tau_q=1$ and $\tau_T=0.3$. The solutions were obtained by numerical evaluation of the integral (2.6). The Fourier transform $\mathcal{F}T(\xi,t)$, Eq. (2.5), decays