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Finite element approximation of the hyperbolic wave equation in mixed form

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Abstract

The purpose of this paper is to present a finite element approximation of the scalar hyperbolic wave equation written in mixed form, that is, introducing an auxiliary vector field to transform the problem into a first-order problem in space and time. We explain why the standard Galerkin method is inappropriate to solve this problem, and propose as alternative a stabilized finite element method that can be cast in the variational multiscale framework. The unknown is split into its finite element component and a remainder, referred to as subscale. As original features of our approach, we consider the possibility of letting the subscales to be time dependent and orthogonal to the finite element space. The formulation depends on algorithmic parameters whose expression is proposed from a heuristic Fourier analysis.

Keywords: Hyperbolic wave equation; Stabilized finite element methods; Orthogonal subscales

1. Introduction

In most engineering problems dealing with waves, the wave equation is written in irreducible form, that is, with a single scalar unknown η depending on the spatial variable $\boldsymbol{x}$ and time t, so that if c is the wave speed this equation reads

$$\frac{1}{c^2}\partial_{tt}^2\eta - \Delta\eta = f, \tag{1}$$

where $\partial_{tt}^2 \equiv \partial_t\partial_t$ is the second order time derivative, Δ is the Laplacian operator and f is a given forcing term. This equation needs to be solved in a spatial domain $\Omega \subset \mathbb{R}^d$ ($d = 1$, 2 or 3) with appropriate boundary conditions and in a time interval $[0, T]$, giving $\eta(\boldsymbol{x}, 0)$ and $\partial_t\eta(\boldsymbol{x}, 0)$ as initial conditions.

However, in some cases it is convenient to consider the mixed form of (1), which consist in solving for η *as well as for a vector function* $\boldsymbol{u}(\boldsymbol{x}, t)$ the problem

$$\mu_\eta\partial_t\eta + \nabla\cdot\boldsymbol{u} = f_\eta, \tag{2}$$

$$\mu_u\partial_t\boldsymbol{u} + \nabla\eta = \boldsymbol{f}_u, \tag{3}$$

where $\mu_\eta > 0$ and $\mu_u > 0$ are coefficients such that $c^2 = (\mu_\eta\mu_u)^{-1}$ and the forcing terms f_η and $\boldsymbol{f}_u$ must be such that $\mu_u\partial_t f_\eta - \nabla\cdot\boldsymbol{f}_u = f$. Another possibility to transform (1) into a first order system is to define $\xi = \partial_t\eta$ and $\boldsymbol{u} = \nabla\eta$. This is the natural option in elastodynamics [4], although in many other physical applications, such as acoustic waves or gravity waves in fluids, the original problem is in fact (2)–(3) and its irreducible form (1). We shall briefly discuss an example of each situation in the following section. It is clear however that the linear problem (2)–(3) is only a model for more involved situations, either in solid mechanics or in nonlinear waves in shallow waters, cases in which the mixed form is mandatory.

The differential operator in (1) is of second order in both space and time, whereas (2)–(3) is a first order evolution problem with first order spatial derivatives. The most popular approach to deal with (1) is to use the Galerkin method for the spatial discretization and then to integrate in time using a finite difference scheme (see for example

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[15]). Numerical difficulties are considered to be concentrated on the time integration scheme, giving for granted that the Galerkin method is optimal for the spatial discretization of the Laplace operator. Therefore, research has been focused on devising time integration schemes for (1) (or vector counterparts) with given design properties.

Much less attention has been paid to the mixed form (2) and (3) of the wave problem. As in most mixed problems, there is a compatibility condition between the interpolation spaces for η and $\boldsymbol{u}$ which can be expressed as an inf–sup condition (see [5] for background, for example). As mentioned in Section 3, this condition on the interpolation of η and $\boldsymbol{u}$ is in fact similar to the condition for pressure and velocity in the case of the Stokes problem or the Darcy problem, but even if it is satisfied it does not guarantee stability of η in the space where it must belong. Our objective will be to present a formulation allowing equal and continuous interpolation for η and $\boldsymbol{u}$. Apart from simplifying the numerical implementation, this has two additional benefits. On the one hand, we will show that improved stability on η can be obtained with respect to some classical methods that satisfy the inf–sup condition. On the other hand, the classical Lagrange interpolation naturally allows for mass lumping through the use of special quadrature rules, a requirement to design explicit time integration schemes and not always possible using some interpolations satisfying the inf–sup condition (see, e.g., [4] and references therein).

Our formulation is based on the variational multiscale approach in the format introduced in [16,17]. The basic idea is to split the unknowns into a *resolvable* component, which can be reproduced by the discretization method (in our case finite elements) and the remainder, which we will call *sub-grid scale* or *subscale*. Rather than solving exactly for the latter, the formulation results from a closed form approximation for the subscales, which is designed in order to capture their *effect* on the discrete finite element solution. This leads to a formulation that allows the use of equal η–$\boldsymbol{u}$ interpolations. We prove analytically this fact in a particular case, only aiming to explain the stabilization mechanism introduced by the approximation of the subscales.

This paper is organized as follows. In Section 2, we state the initial and boundary value problem to be solved, both in its differential and in its weak form. We also present two examples of wave problems that will serve us to illustrate a discussion on the way to scale the equations. The Galerkin space discretization is presented in Section 3, where the reasons for its failure are explained. The main contribution of this work is presented in Section 4, where a stabilized finite element method is proposed. After presenting the basis of the formulation, its application to the mixed form of the wave equation is studied in detail. The algorithmic parameters on which the formulation depends are designed on the basis of a Fourier analysis of the problem, similar to that proposed already in [8] for the incompressible Navier–Stokes equations, although extended to general first-order systems. A stability estimate is then proved in the particular case in which the space of subcales is orthogonal to the finite element space, a possibility introduced in [10] to stabilize velocity–pressure interpolations in the Stokes problem (see also [9] for a full analysis of the method applied to the linearized Navier–Stokes equations). Section 5 presents the results of some numerical experiments only intended to demonstrate that the stabilized formulation proposed in fact suppresses the instabilities of the Galerkin method. Some concluding remarks close the paper.

2. Problem statement

2.1. Initial and boundary value problem

The differential equations (2) and (3) need to be supplied with adequate initial and boundary condition to define the problem to be solved.

As for the boundary conditions, we consider two possibilities. Let the boundary $\partial\Omega$ be split into two disjoint sets Γ_{I} and Γ_{R}. On the former, we consider prescribed the scalar η and on Γ_{R} the normal component of vector $\boldsymbol{u}$ is assumed to be given. Without loss of generality, we will consider both boundary conditions as homogeneous. The initial conditions to be considered are of the form $\eta(\boldsymbol{x},0)=\eta^0(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}^0(\boldsymbol{x})$.

The differential form of the initial and boundary value problem to be considered consists therefore in finding $\eta(\boldsymbol{x},t)$ and $\boldsymbol{u}(\boldsymbol{x},t)$ such that

$$\mu_\eta \partial_t \eta + \nabla\cdot\boldsymbol{u} = f_\eta, \quad \text{in } \Omega,\ t>0, \tag{4}$$

$$\mu_u \partial_t \boldsymbol{u} + \nabla\eta = \boldsymbol{f}_u, \quad \text{in } \Omega,\ t>0, \tag{5}$$

$$\eta = 0, \quad \text{on } \Gamma_{\mathrm{I}},\ t>0, \tag{6}$$

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0, \quad \text{on } \Gamma_{\mathrm{R}},\ t>0, \tag{7}$$

$$\eta(\boldsymbol{x},0) = \eta^0(\boldsymbol{x}), \quad \text{in } \Omega, \tag{8}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad \text{in } \Omega. \tag{9}$$

Eqs. (4) and (5) can be re-written as

$$\begin{bmatrix} \mu_\eta & \boldsymbol{0} \\ \boldsymbol{0} & \mu_u \boldsymbol{I} \end{bmatrix} \partial_t \begin{bmatrix} \eta \\ \boldsymbol{u} \end{bmatrix} + \begin{bmatrix} 0 & \nabla\cdot(\cdot) \\ \nabla(\cdot) & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \eta \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} f_\eta \\ \boldsymbol{f}_u \end{bmatrix},$$

where $\boldsymbol{I}$ is the $d\times d$ identity matrix.

At this point it is convenient to introduce some notation. Given X, a space of functions defined on Ω, its norm will be denoted by $\|\cdot\|_X$, and the space of functions such that their X-norm is C^k continuous in the time interval $[0,T]$ will be denoted by $C^k([0,T];X)$. We will be interested only in the cases $k=0$ and $k=1$. Three particular spaces X will be relevant in the presentation: $L^2(\Omega)$, $H^1(\Omega)$, the space of functions in $L^2(\Omega)$ with derivatives also in $L^2(\Omega)$, and $H(\mathrm{div},\Omega)$, the space of vector functions with components and divergence in $L^2(\Omega)$. A bold character will be used to denote the vector counterpart of the first two spaces.

As it will be explained below, for regular enough data the problem is well posed for $\eta\in C^0([0,T],V_\eta)\cap C^1([0,T],L^2(\Omega))$ and $\boldsymbol{u}\in C^0([0,T],\boldsymbol{V}_u)\cap C^1([0,T],\boldsymbol{L}^2(\Omega))$, where