Journal of Non-Newtonian Fluid Mechanics 249 (2017) 48–52

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: www.elsevier.com/locate/jnnfm

Short Communication

Transient computations using the natural stress formulation for solving sharp corner flows

J.D. Evans [a,*], C.M. Oishi [b]

[a] Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, United Kingdom
[b] Departamento de Matemática e Computação, Faculdade de Ciências e Tecnologia, Universidade Estadual Paulista "Júlio de Mesquita Filho" 19060-900 Presidente Prudente, Sao Paulo, Brazil

ARTICLE INFO

Article history:
Received 26 May 2017
Revised 24 August 2017
Accepted 30 August 2017
Available online 19 September 2017

Keywords:
Unsteady viscoelastic flows
Oldroyd-B model
Natural stress formulation
Planar contraction
Sharp corner

ABSTRACT

In this short communication, we analyse the potential of the natural stress formulation (NSF) (i.e. aligning the stress basis along streamlines) for computing planar flows of an Oldroyd-B fluid around sharp corners. This is the first attempt to combine the NSF into a numerical strategy for solving a transient fluid flow problem considering the momentum equation in Navier–Stokes form (the elastic stress entering as a source term) and using the constitutive equations for natural stress variables. Preliminary results of the NSF are motivating in the sense that accuracy of the numerical solution for the extra stress tensor is improved near to the sharp corner. Comparison studies among the NSF and the Cartesian stress formulation (CSF) (i.e. using a fixed Cartesian stress basis) are conducted in a typical benchmark viscoelastic fluid flow involving a sharp corner: the 4 : 1 contraction. The CSF needs a mesh approximately 10 times smaller to capture similar near singularity results to the NSF.

1. Introduction

Flow through a contraction is a benchmark problem in computational rheology [9], where viscoelastic fluids exhibit regions of strong shearing near the walls and uniaxial extension along the centreline. The complex flow patterns that evolve have been the subject of much interest in the literature, with attention focused on: (i) vortex behaviour, both near the re-entrant corner (so-called lip vortices) and salient corners, (ii) variation of the pressure drops across the contraction with strength of fluid elasticity (Weissenberg number), (iii) particle paths upstream of the contraction and (iv) velocity overshoots along the axis of symmetry.

The main numerical approaches to simulate this flow, have been finite-difference [21], finite-element [8] and finite-volume [2]. Combinations of the methods, e.g. hybrid finite-element finite-volume [1] as well as Lagrangian and semi-Langrangian methods [15,22] are also commonly employed. However, a key feature of all numerical schemes so far employed is that they discretize the viscoelastic constitutive equations formulated using a fixed Cartesian basis for the stress. We refer to this formulation as the Cartesian stress formulation (CSF) of the constitutive equations. An alternative approach is to exploit the mathematical structure of the upper convected derivative and align the stress basis with the flow using streamlines. This formulation uses the velocity field to span the stress field, the formulation of the constitutive equations in this setting being termed the natural stress formulation (NSF).

The natural stress formulation was first used by Renardy [17], to demonstrate its ability to eliminate the downstream stress instability encountered during numerical integration around re-entrant corners [18]. Although the idea of transforming the stress tensor components to a basis aligned with streamlines for computation purposes had been recognised previously in [3,11]. However, the full power of the approach has not yet been exploited numerically in a mathematically systematic way for the full contraction geometry.

A key feature of the geometry is the presence of the re-entrant corner at which the velocity gradients and stress are infinite. The singularity determination for the Oldroyd-B fluid was first given by Hinch [10], with the asymptotic structure of the local solution completed through the upstream wall boundary layer by Renardy [19] and the boundary layer at the downstream wall in [16] and [4,5]. An important aspect of the solution analysis is that the natural stress formulation is an efficient way to transmit the necessary stress information from the upstream to downstream regions. This is explicitly calculated in [6,7] for the UCM fluid, illustrating that the necessary stress information is contained in high-order terms of the asymptotic expansions when using the Cartesian stress com-

* Corresponding author.
E-mail addresses: masjde@maths.bath.ac.uk, masjde@bath.ac.uk (J.D. Evans), oishi@fct.unesp.br (C.M. Oishi).

https://doi.org/10.1016/j.jnnfm.2017.08.012
0377-0257/

ponents, with the natural stress variables being able to uncouple and extract this information.

The success of the natural stress formulation near the re-entrant corner singularity, both for asymptotics and numerical computation, are the motivating reasons to investigate the formulation for the full contraction geometry.

2. Mathematical formulation

The governing equations for the incompressible flow of a viscoelastic fluid we adopt are the continuity, momentum and Oldroyd-B constitutive equations in dimensionless form

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\mathrm{Re}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \beta\nabla^2\mathbf{v} + \nabla\cdot\mathbf{T}, \tag{2}$$

$$\mathbf{T} + \mathrm{Wi}\left(\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T\right) = 2(1-\beta)\mathbf{D}. \tag{3}$$

Here $\mathbf{v}$ is the velocity field, p is an arbitrary isotropic pressure, $\mathbf{T}$ is the polymeric contribution to the extra-stress tensor and $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ is the rate-of-strain tensor. The dimensionless parameters are the Reynolds number Re, Weissenberg number Wi and retardation parameter $\beta \in [0, 1]$ (the dimensionless retardation time or solvent viscosity ratio) defined by

$$\mathrm{Re} = \frac{\rho UL}{\eta_s + \eta_p}, \quad \mathrm{Wi} = \frac{\lambda_1 U}{L}, \quad \beta = \frac{\eta_s}{\eta_s + \eta_p}, \tag{4}$$

where ρ is the density, U and L are characteristic length and flow speeds respectively, λ_1 the relaxation time, η_s the solvent viscosity and η_p the polymer viscosity. The above governing equations have been made non-dimensional using L for the spatial variables, U for the velocity scaling and $(\eta_p + \eta_s)U/L$ the pressure and stress scalings. The total stress is $\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}$, with the extra-stress tensor $\boldsymbol{\tau} = \mathbf{T} + 2\beta\mathbf{D}$ being rheologically composed of polymer and Newtonian solvent contributions.

2.1. Cartesian stress formulation

Denoting by $\mathbf{i}$ and $\mathbf{j}$ the unit vectors in fixed Cartesian x and y directions, we have

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} = (u, v)^T \tag{5}$$

and

$$\mathbf{T} = T_{11}\mathbf{i}\mathbf{i}^T + T_{12}\left(\mathbf{i}\mathbf{j}^T + \mathbf{j}\mathbf{i}^T\right) + T_{22}\mathbf{j}\mathbf{j}^T. \tag{6}$$

The component form of the polymer constitutive Eq. (3) for the Cartesian extra-stresses T_{11}, T_{12}, T_{22} is

$$\begin{aligned}
&T_{11} + \mathrm{Wi}\left(\frac{\partial T_{11}}{\partial t} + u\frac{\partial T_{11}}{\partial x} + v\frac{\partial T_{11}}{\partial y} - 2\frac{\partial u}{\partial x}T_{11} - 2\frac{\partial u}{\partial y}T_{12}\right)\\
&\quad = 2(1-\beta)\frac{\partial u}{\partial x},\\
&T_{22} + \mathrm{Wi}\left(\frac{\partial T_{22}}{\partial t} + u\frac{\partial T_{22}}{\partial x} + v\frac{\partial T_{22}}{\partial y} - 2\frac{\partial v}{\partial y}T_{22} - 2\frac{\partial v}{\partial x}T_{12}\right)\\
&\quad = 2(1-\beta)\frac{\partial v}{\partial y},\\
&T_{12} + \mathrm{Wi}\left(\frac{\partial T_{12}}{\partial t} + u\frac{\partial T_{12}}{\partial x} + v\frac{\partial T_{12}}{\partial y} - \frac{\partial v}{\partial x}T_{11} - \frac{\partial u}{\partial y}T_{22}\right)\\
&\quad = (1-\beta)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).
\end{aligned} \tag{7}$$

2.2. Natural stress formulation

Aligning the polymer stress basis along streamlines, introduces the so called natural stress variables. We follow the construction of Renardy [17] (see also [20,23]). Introducing the configuration tensor $\mathbf{A}$ by

$$\mathbf{T} = \frac{(1-\beta)}{\mathrm{Wi}}(\mathbf{A} - \mathbf{I}), \tag{8}$$

the polymer constitutive Eq. (3) becomes

$$\left[\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} - (\nabla\mathbf{v})\mathbf{A} - \mathbf{A}(\nabla\mathbf{v})^T\right] + \frac{1}{\mathrm{Wi}}(\mathbf{A} - \mathbf{I}) = 0. \tag{9}$$

We now express $\mathbf{A}$ in terms of the dyadic products of the velocity $\mathbf{v}$ and an orthogonal vector $\mathbf{w}$ defined as follows

$$\mathbf{w} = \frac{1}{|\mathbf{v}|^2}(-v, u)^T,$$

so that

$$\mathbf{A} = \lambda\mathbf{v}\mathbf{v}^T + \mu(\mathbf{v}\mathbf{w}^T + \mathbf{w}\mathbf{v}^T) + \nu\mathbf{w}\mathbf{w}^T, \tag{10}$$

where λ, μ, ν are termed the natural stress variables. However, a detraction of this construction is that the basis vectors $\mathbf{v}, \mathbf{w}$ are degenerate when the velocity field vanishes. As such, it is convenient to use instead, unit vectors in their directions. Hence, we write

$$\mathbf{A} = \hat{\lambda}\hat{\mathbf{v}}\hat{\mathbf{v}}^T + \hat{\mu}(\hat{\mathbf{v}}\hat{\mathbf{w}}^T + \hat{\mathbf{w}}\hat{\mathbf{v}}^T) + \hat{\nu}\hat{\mathbf{w}}\hat{\mathbf{w}}^T, \tag{11}$$

where

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad \hat{\mathbf{w}} = |\mathbf{v}|\mathbf{w}, \quad \hat{\lambda} = |\mathbf{v}|^2\lambda, \quad \hat{\mu} = \mu, \quad \hat{\nu} = \frac{\nu}{|\mathbf{v}|^2}. \tag{12}$$

The scaled natural stress variables $\hat{\lambda}, \hat{\mu}, \hat{\nu}$ then satisfy the component equations

$$\begin{aligned}
&\left[\frac{\partial\hat{\lambda}}{\partial t} + \frac{2\hat{\mu}}{|\mathbf{v}|^2}\left(v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\right) + |\mathbf{v}|^2(\mathbf{v}\cdot\nabla)\left(\frac{\hat{\lambda}}{|\mathbf{v}|^2}\right) + 2\hat{\mu}|\mathbf{v}|^2\nabla\cdot\mathbf{w}\right]\\
&\quad + \frac{1}{\mathrm{Wi}}\left(\hat{\lambda} - 1\right) = 0,\\
&\left[\frac{\partial\hat{\mu}}{\partial t} + \left(\frac{\hat{\lambda} - \hat{\nu}}{|\mathbf{v}|^2}\right)\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) + (\mathbf{v}\cdot\nabla)\hat{\mu} + \hat{\nu}|\mathbf{v}|^2\nabla\cdot\mathbf{w}\right]\\
&\quad + \frac{\hat{\mu}}{\mathrm{Wi}} = 0,\\
&\left[\frac{\partial\hat{\nu}}{\partial t} + \frac{2\hat{\mu}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) + \frac{1}{|\mathbf{v}|^2}(\mathbf{v}\cdot\nabla)\left(\hat{\nu}|\mathbf{v}|^2\right)\right]\\
&\quad + \frac{1}{\mathrm{Wi}}\left(\hat{\nu} - 1\right) = 0,
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
|\mathbf{v}|^2\nabla.\mathbf{w} &= |\mathbf{v}|^2\left(\frac{\partial}{\partial x}\left(-\frac{v}{|\mathbf{v}|^2}\right) + \frac{\partial}{\partial y}\left(\frac{u}{|\mathbf{v}|^2}\right)\right)\\
&= \frac{1}{|\mathbf{v}|^2}\left((v^2 - u^2)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 4uv\frac{\partial u}{\partial x}\right).
\end{aligned}$$

The component form of (8) is

$$\begin{aligned}
T_{11} &= \frac{(1-\beta)}{\mathrm{Wi}}\left(-1 + \frac{1}{|\mathbf{v}|^2}\left(\hat{\lambda}u^2 - 2\hat{\mu}uv + \hat{\nu}v^2\right)\right),\\
T_{12} &= \frac{(1-\beta)}{\mathrm{Wi}|\mathbf{v}|^2}\left(\hat{\lambda}uv + \hat{\mu}\left(u^2 - v^2\right) - \hat{\nu}uv\right),\\
T_{22} &= \frac{(1-\beta)}{\mathrm{Wi}}\left(-1 + \frac{1}{|\mathbf{v}|^2}\left(\hat{\lambda}v^2 + 2\hat{\mu}uv + \hat{\nu}u^2\right)\right),
\end{aligned} \tag{14}$$