



Stokes' third problem for Herschel–Bulkley fluids



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ARTICLE INFO

Article history:

Received 2 August 2016

Revised 18 March 2017

Accepted 25 March 2017

Available online 27 March 2017

Keywords:

Herschel–Bulkley fluids

Stokes problem

Lubrication theory

Shear flow

Depth-averaged equations

ABSTRACT

Herschel–Bulkley materials can be set in motion when a sufficiently high shear stress or body force is applied to them. We investigate the behaviour of a layer of Herschel–Bulkley fluid when it is suddenly tilted and subject to gravitational forces. The material's dynamic response depends on the details of its constitutive equation. When its rheological behaviour is viscoelastoplastic with no thixotropic behaviour, the material is set in motion instantaneously along its entire base. When its rheological behaviour involves two yield stresses (static and dynamic yield stresses), the material must be destabilised before it starts to flow. This problem is thus similar to a Stefan problem, with an interface that separates the sheared and unsheared regions and moves from top to bottom. We estimate the time needed to set the layer in motion in both cases. We also compare the solution to the local balance equations with the solution to the depth-averaged mass and momentum equations and show that the latter does not provide consistent solutions for this flow geometry.

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1. Introduction

Viscoplastic fluid theory has long been used to approximate the complex rheological behaviour of natural materials such as snow and mud, particularly their transition between solid- and fluid-like states [1]. The theory's strength lies in its capacity to describe flow initiation and cessation using a single constitutive equation. Natural materials can also entrain the bed on which they flow and, in this case, it is tempting to see basal entrainment as a form of yielding induced by the passage of the flow [2–4].

Various processes are at work when bed materials are set in motion. Among these, two are expected to play a major part: the increase in the normal and shear stresses applied to the bed surface, and the decrease in the shear strength relative to gravitational forces. The first process is certainly the easiest to investigate experimentally and theoretically. The Stokes problem provides a theoretical perspective: fluid is set in motion by applying a shear stress to its boundary or by moving that boundary at a constant velocity [5,6]. The second process can be studied by suddenly applying a body force to the fluid initially at rest. For convenience, this paper refers to this problem as Stokes' third problem. For Newtonian fluids, there exists a similarity solution to this problem, which shows that the fluid is instantaneously set in motion and virtually all of the fluid layer is entrained even though the effects far

from the boundary are exponentially small [6]. Herschel–Bulkley materials display a more complex dynamic response to a sudden change in the stress state than do Newtonian fluids. This is because of their ability to remain static when the stress state lies below a certain threshold, although they yield when the stress state moves above it. This paper investigates Stokes' third problem for Herschel–Bulkley fluids.

The key issue in Stokes' first and third problems is the existence of an interface separating the yielded and unyielded flows. If this interface exists, then one should be able to determine its propagation velocity and, thereby, the entrainment rate (at least in ideal cases, such as Stokes' problems). For Stokes' first problem and classic Herschel–Bulkley materials, there is no interface and the material is set in motion instantaneously over its whole depth [7,8]. For Stokes' third problem and Herschel–Bulkley materials exhibiting thixotropy, recent studies have posited the existence of interfaces moving at constant velocity [2,3], but the formal proof is lacking.

The problem of determining entrainment rates has also been addressed within the framework of depth-averaged equations (see [9] for a review). As the mass and momentum balance equations are averaged, the interface between sheared and unsheared flows is systematically treated as a shock wave (its propagation velocity must satisfy the Rankine–Hugoniot equation regardless of the constitutive equation, see Section 2.1). Although depth-averaging leads to governing equations that are simpler to solve, they are not closed. The governing equations must be supplemented by closure equations that specify how local variables (such as the bottom

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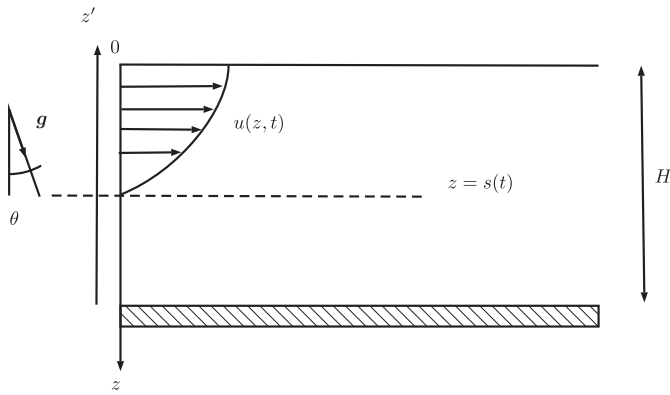


Fig. 1. Setting in motion a volume of fluid suddenly tilted at an angle θ .

shear stress and the entrainment rate) are related to bulk quantities (such as the depth-averaged velocity and flow depth). To date, most closure equations for non-Newtonian fluids have been based on empirical considerations and thus lack consensus [9].

This paper's objective is to explore the possibility of fluid-solid interfaces for Stokes's third problem and Herschel–Bulkley fluids. It is the continuation of previous studies devoted to Stokes' first [7,8] and second [10] problems. We begin by setting out what we refer to as Stokes' third problem (Section 2). We focus on Herschel–Bulkley fluids and outline the current state of the art in modelling Herschel–Bulkley fluids. The paper strays from the classic form of the Herschel–Bulkley constitutive equation in order to take advantage of recent developments in the rheometrical investigation of viscoplastic materials. Indeed, the classic form assumes that the material behaves like a rigid body when the stress state is below a given threshold, whereas in basal entrainment problems we expect the material's behaviour in its solid state to affect the entrainment dynamics. Our literature review led us to consider two types of Herschel–Bulkley fluids: simple Herschel–Bulkley fluids, whose rheological behaviour is well described by a one-to-one constitutive equation, and non-simple Herschel–Bulkley fluids, whose rheological behaviour exhibits shear-history dependence. We demonstrate that the details of the constitutive equation have a great deal of influence on the solution to Stokes' third problem. In Section 3, which is devoted to simple Herschel–Bulkley fluids, we show that the material is set in motion instantaneously. By contrast, non-simple Herschel–Bulkley materials do not start moving spontaneously; they must first be destabilised. A front subsequently propagates through the static layer and sets it in motion (Section 4). For non-simple Herschel–Bulkley materials, we also show that in the absence of slip, the depth-averaged equations do not require a closure equation for the entrainment rate, but the solution to these equations is physically inconsistent.

2. Stokes' third problem

The literature refers to two Stokes problems. Stokes' first problem refers to the impulsive motion of a semi-infinite volume of Newtonian fluid sheared by an infinite solid boundary. Stokes' second problem concerns the cyclical motion of this volume sheared by an oscillatory boundary [6]. These two problems have also been investigated for viscoplastic materials [7,8,10].

A related problem concerns the setting in motion of a layer of fluid of depth H , initially at rest and suddenly tilted at an angle θ to the horizontal (see Fig. 1). Contrary to the two Stokes problems above, we consider a volume that is not bounded by an infinite plate, but by a free surface. As this problem bears some resemblance to the original Stokes problem, this paper refers to it as Stokes' third problem (mainly for convenience). Previously, it

was partially studied for Herschel–Bulkley flows [2,3] and Drucker–Prager fluid [4].

2.1. Governing equations

We consider an incompressible Herschel–Bulkley fluid with density ρ ; its constitutive equation is discussed in Section 2.2. The fluid is initially at rest. There is a free surface located at $z = 0$, with the z -axis normal to the free surface and pointing downward. We also introduce the z' -axis, normal to the free surface, but pointing upward. The x -axis is parallel to the free surface. At time $t = 0$, the volume is instantaneously tilted at an angle θ to the horizontal. We assume that a simple shear flow takes place under the effects of gravitational forces and that the flow is invariant under any translation in the x -direction. The initial velocity is

$$u(z, 0) = 0. \quad (1)$$

At the free surface $z = 0$, in the absence of traction, the shear stress τ is zero

$$\tau = 0 \text{ at } z = 0. \quad (2)$$

A key issue in Stokes' third problem is the existence of a propagation front $z = s(t)$ (i.e. a moving interface between the sheared and stationary layers) and the boundary conditions at this front. For Stokes' first problem, shear-thinning viscoplastic fluids behave like Newtonian fluids: the momentum balance equation reduces to a linear parabolic equation, and the front propagates downward instantaneously [7,8]. The question arises as to whether this is also the case for Stokes' third problem.

Let us admit that the interface moves at a finite velocity v_f . The dynamic boundary condition at this interface is given by a Rankine–Hugoniot equation

$$[-\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n} - v_f) + \boldsymbol{\sigma} \cdot \mathbf{n}] = 0, \quad (3)$$

where $[[f]]$ denotes f 's jump across the interface [11,12]. In the absence of slip

$$\mathbf{u} = 0 \text{ at } z = s(t), \quad (4)$$

this equation implies the continuity of the stresses across the interface

$$[[\tau]] = 0 \text{ and } [[\sigma_{zz}]] = 0, \quad (5)$$

where σ_{zz} is the normal stress in the z -direction. If the material slips along the bed-flow interface at a velocity u_s , then the Rankine–Hugoniot equation implies that the shear stress exhibits a jump across the interface, while the normal stress is continuous

$$[[\tau]] = -\rho u_s v_f \text{ and } [[\sigma_{zz}]] = 0.$$

The first relationship has often been used in the form $v_f = -[[\tau]]/(\rho u_s)$, which fixes the entrainment rate when the other variables are prescribed [3,13,14]. Internal slip in viscoplastic materials is only partially understood. It may be a consequence of shear localisation or shear banding in thixotropic viscoplastic fluids [15,16]. In the rest of the paper, we assume that the no-slip condition applies at the interface, and so the boundary condition is given by equation (5).

For this problem, the governing equation is derived from the momentum balance equation in the x -direction

$$\rho \frac{\partial u}{\partial t} = \rho g \sin \theta - \frac{\partial \tau}{\partial z}. \quad (6)$$

To solve the initial boundary value problem (2)–(6), we need to specify the constitutive equation.

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