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Viscoplastic dimensionless numbers

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ABSTRACT

In the present paper we analyze the dimensionless numbers that concern the flow of viscoplastic materials. The Bingham material is used to conduct the main discussion but the ideas are generalized to more complex viscoplastic models at the end of the article. Although one can explore the space of solutions with a set of dimensionless numbers where only one of them takes into account the yield stress, like the Bingham number for example, we recommend that the characteristic stress should be defined as the extra-stress intensity evaluated at a characteristic (maximum) deformation rate. Such a definition includes the yield stress in every dimensionless number that is related to viscous effects like the Reynolds number, the viscosity ratio, and the Rayleigh number. This procedure was shown to be more effective on collapsing data into master curves and to provide a fairer comparison with the Newtonian case. This happens because a more representative viscous effect is taken into account, concentrating the plastic effects into a single parameter. The plastic number, the ratio of the yield stress to the maximum stress of the domain, is shown to better capture plastic effects than the usual Bingham number. The analysis of problems where a characteristic stress, but not a characteristic velocity, is provided, indicates that a more representative characteristic velocity should be defined with respect to the driving potential for the motion, i.e., the difference between the characteristic and yield stresses. This method is in contrast to the majority of the literature, where for Bingham materials, the dimensionless numbers are maintained in the same form as the original Newtonian ones, replacing the Newtonian viscosity by the viscous parameter of the Bingham model.

1. Introduction

Dimensionless analysis is a common practice in Fluid Mechanics and other areas of research. Dimensionless numbers are able to condense information about the physics of the problem, and therefore its usefulness is undisputed. This approach allows a reduction in the number of experiments, numerical or not, in order to understand the role played by the different parameters. In addition, when a specific dimensionless number tends to zero or to infinity, generally a certain term (probably associated with a certain phenomenon) is negligible, which in turn enables the establishment of limiting cases.

The Buckinham-π theorem is an important tool that provides the minimum number of dimensionless quantities necessary for the complete description of the flow problem, assuming a given set of dimensional parameters that are connected to the physics of the problem. Although from a mathematical viewpoint one is able to cover the whole space of possible dynamic responses of the system of equations, the Buckinham-π procedure does not offer a unique set of dimensionless numbers. Therefore, a natural issue that arises concerns the comparison of the different possibilities. Is there a preferable set of choices, or is any choice that covers the space of possibilities equally legitimate or useful?

Informal discussions of this subject, during the VFTA2015 congress,[1] have brought out at least three different answers to this question.

(1) There are no conceptual differences between the different possible sets.
(2) Depending on the problem, some sets can be more useful.
(3) Irrespectively of the problem, some sets are more useful.

Since there is no consensus on this issue, a discussion of the subject seems to be worth making. Analyses on the

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dimensionless numbers of non-Newtonian fluids in the literature are more active on the flow of viscoelastic fluids. It is not the aim of the present work to provide a historical treatment on the subject, but important discussions on the use of *Deborah* and *Weissenberg* numbers can be found in [1–6] and, more recently in [7,8]. The reader is referred to those articles for a broad viewpoint on this matter. Another contribution to dimensionless numbers in non-Newtonian fluid mechanics was made by [9].

When dealing with the flow of non-Newtonian fluids, an important observation lies on the fact that this class of materials is defined by exclusion, i.e., a non-Newtonian effect is, by definition, an effect which cannot be Newtonian. In this sense, the Newtonian fluid is clearly a reference. The logical consequence is that, whenever possible, it would be interesting to consider the Newtonian flow as the limiting case of a non-Newtonian one. In this connection, Weissenberg and Deborah numbers have been conceived as viscoelastic dimensionless numbers that vanish in the Newtonian case. In the same spirit, the Bingham number is a dimensionless viscoplastic number that is also zero in the Newtonian case.

While the importance of defining a dimensionless number that accounts for a strictly non-Newtonian effect is recognized and highlighted, less attention is devoted to dimensionless numbers which also appear in the flow of Newtonian fluids. Quantities like the Reynolds number, the viscosity ratio, and the capillary number, among others, are based on the competition between viscous effects and other phenomena: inertia, viscous effects of a second fluid, interfacial tension (to give the corresponding examples). Since the viscosity behavior of a non-Newtonian material differs, in general, from the Newtonian one, careful consideration is needed if one wishes to maintain in the non-Newtonian problem the interpretation of these dimensionless numbers conceived in a Newtonian paradigm. This principle is consistent with the logic behind the maintenance of the Newtonian nomenclature in a non-Newtonian problem.

At this point, it is worth noticing that there are indeed examples of rigor with respect to this issue. The Reynolds number expression for the flow of a power-law fluid, $\eta = K\dot{\gamma}^{n-1}$, in a tube of diameter D with mean velocity V can be found in pioneering papers, e.g. [10], and is given by

$$Re = \frac{8n}{6n+2}\frac{\rho V^{2-n}D^n}{K}. \tag{1}$$

We can notice that Eq. (1) is not straightforward at first glance. The rationale that leads to Eq. (1) is based on the following general definition.

$$Re = \frac{8\rho U^2}{\tau_w}, \tag{2}$$

which, combined with the definition for the friction factor, $f = 4\tau_w/0.5\rho U,^2$leads to a constant product of the friction factor with the Reynolds number for laminar flows: $fRe = 64$. In fact, any definition that intends to keep the original meaning of each of these quantities must lead to the constancy of the product *fRe*. This point of view was adopted by [11] in the context of a viscoplastic problem. In [12], these definitions for the Reynolds number and friction factor were used to find expressions for the localized head loss for fittings in pipes for power-law and viscoplastic materials.

This example illustrates situations where a more careful construction of the dimensionless numbers that have a Newtonian counterpart leads to more useful quantities. The purpose of the present paper is to investigate the use of dimensionless numbers in studying the flows of viscoplastic materials.

2. Detachment from the purely viscous behavior

We will first approach this matter by considering Bingham-like materials and then extend this to other viscoplastic constitutive equations. The Bingham model can be expressed in two forms: a *stress* form and a *viscosity* form, respectively given by

$$\begin{cases} \tau = \tau_y + \mu_B\dot{\gamma} & \text{if} \quad \tau \geq \tau_y \\ \dot{\gamma} = 0 & \text{if} \quad \tau < \tau_y \end{cases} \tag{3}$$

and

$$\begin{cases} \eta = \dfrac{\tau_y}{\dot{\gamma}} + \mu_B & \text{if} \quad \tau \geq \tau_y \\ \dot{\gamma} = 0 & \text{if} \quad \tau < \tau_y \end{cases} \tag{4}$$

where $\tau = \sqrt{0.5\,\mathrm{tr}(\boldsymbol{\tau}^2)}$ is the extra-stress intensity and the extra-stress tensor is written as a Generalized Newtonian Fluid, i.e., $\boldsymbol{\tau} = \eta\dot{\boldsymbol{\gamma}}$, where $\dot{\boldsymbol{\gamma}}$ is twice the rate-of-strain tensor. The quantity τ_y is the yield stress, while $\dot{\gamma} = \sqrt{0.5\,\mathrm{tr}(\dot{\boldsymbol{\gamma}}^2)}$ is the deformation rate. Interestingly, the expression *the viscosity of a Bingham material* is ambiguous. One can be referring to μ_B or η. In the present paper we will distinguish between the two by referring to η as the viscosity of the Bingham material or simply "the viscosity,"[2] while μ_B will be referred to as the viscous parameter of the Bingham material.[3]

It is a common practice in viscoplastic problems of Bingham materials to consider the characteristic stress of the material as $\tau_a = \mu_B\dot{\gamma}_c = \mu_B U/L$, where $\dot{\gamma}_c$ is a characteristic deformation rate, generally expressed by a characteristic velocity and a characteristic length scale. When using the Hershell–Buckley (HB) model, where the term $K\dot{\gamma}^n$ replaces $\mu_B\dot{\gamma}$ in Eq. (3), the usual characteristic stress adopted in the literature is $K(U/L)^n$. In fact, the Bingham number, *Bn*, appears naturally if one divides Eq. (3) by τ_a:

$$\begin{cases} \tau_a^* = Bn + \dot{\gamma}^* & \text{if} \quad \tau_a^* \geq Bn \\ \dot{\gamma}^* = 0 & \text{if} \quad \tau_a^* < Bn \end{cases}, \tag{5}$$

where $\tau_a^* = \tau/\tau_a$ is the dimensionless stress, $\dot{\gamma}^* = \dot{\gamma}L/U$ is the dimensionless deformation rate, and

$$Bn = \frac{\tau_y L}{\mu_B U} \tag{6}$$

is the usual Bingham number.

Here we can connect the issues raised above with the particular case of a Bingham material. If we look at a viscoplastic Bingham problem without any pre-conceived notions we observe that, besides $\tau_a = \mu_B U/L$, there are two other natural candidates for a characteristic stress: $\tau_b = \tau_y + \mu_B U/L$ and simply τ_y. The natural questions that arise at this point are: 1) Why is it usual in the literature to use τ_a?; 2) Does it matter which kind of characteristic stress is used?; in case the answer to the second question is "yes," 3) Is τ_a the best option?

Although one can choose the yield stress as the characteristic stress when the study is confined to viscoplastic materials, this choice has the inconvenient feature of being more difficult to compare with the Newtonian case, since the dimensionless stresses go to infinity in this limit. Hence, we have one alternative which is to use τ_b as a characteristic stress. In this case, Eq. (3) can be rewritten as

$$\begin{cases} \tau_b^* = Pl + (1 - Pl)\dot{\gamma}^* & \text{if} \quad \tau_b^* \geq Pl \\ \dot{\gamma}^* = 0 & \text{if} \quad \tau_b^* < Pl \end{cases}, \tag{7}$$

where $\tau_b^* = \tau/\tau_b$ is a dimensionless stress and the dimensionless quantity *Pl* is called, from now on, the *plastic number* and is defined as

$$Pl = \frac{\tau_y}{\tau_y + \mu_B U/L}. \tag{8}$$

[2] Although this quantity was called "the true viscosity" or the "effective viscosity" in the literature, to call it simply "the viscosity" seems to be more in accordance with the official nomenclature (see [13]).

[3] Although this quantity is generally referred to as *the plastic viscosity*, we are avoiding this expression here for the reason pointed out by [14], viz. the fact that this viscosity parameter is the viscosity of the material at high shear rates, where the role played by the yield stress is negligible.

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