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Flow of a thixotropic or antithixotropic fluid in a slowly varying channel: The weakly advective regime

David Pritchard, Stephen K. Wilson*, Catriona R. McArdle

Department of Mathematics and Statistics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow, G1 1XH, Scotland, United Kingdom

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ABSTRACT

A general formulation of the governing equations for the slow, steady, two-dimensional flow of a thixotropic or antithixotropic fluid in a channel of slowly varying width is described. These equations are equivalent to the equations of classical lubrication theory for a Newtonian fluid, but incorporate the evolving microstructure of the fluid, described in terms of a scalar structure parameter. We demonstrate how the lubrication equations can be further simplified in the weakly advective regime in which the advective Deborah number is comparable to the aspect ratio of the flow, and present illustrative analytical and semi-analytical solutions for particular choices of the constitutive and kinetic laws, including a purely viscous Moore–Mewis–Wagner model and a regularised viscoplastic Houška model. The lubrication results also allow the calibration and validation of cross-sectionally averaged, or otherwise reduced, descriptions of thixotropic channel flow which provide a first step towards models of thixotropic flow in porous media, and we employ them to explain why such descriptions may be inadequate.

1. Introduction

Recent years have seen increasing interest in thixotropic flow. This interest stems both from applications, which include the flow of muds, processed foods, polymer solutions and waxy crude oils, and from the challenge that thixotropic fluids present to the modeller. Typically, the macroscopic rheological properties of such a fluid depend on its microscopic structure (for example, a network of flocculated colloidal particles or a tangle of long-chain polymers [1]) and thixotropy arises because the microstructure gradually breaks down under shear and rebuilds through Brownian motion. The theoretical modeller is faced with two problems: the rheometric problem of describing this build-up and breakdown, along with the corresponding relationship between the structure and the rheology; and the fluid-dynamical problem of describing the resulting flows.

Most attention has been paid to the rheometric problem. In the simplest models of thixotropic fluids, the state of the microstructure is described by a scalar "structure parameter" λ, which evolves according to an advection–kinetic equation. Many such models have been developed over the last fifty years and calibrated against rheometric data [1,2]. However, less research has been carried out on non-rheometric flows, and it is still uncertain how thixotropy manifests itself even in many "classical" fluid-dynamical problems.

Lubrication flow is a category of such classical problems. In the lubrication regime, the different streamwise and transverse lengthscales of a flow allow the governing equations to be significantly simplified, and in some problems permit the transverse variation to be averaged out or otherwise eliminated from the problem [3,4]. Classical lubrication theory for a Newtonian fluid was first developed by Reynolds [5], and has since been extended to a number of non-Newtonian fluids. For example, the theory for viscoplastic fluids, first put on a systematic basis by Balmforth and Craster [6] and subsequently extended [7–10], has been applied to the flow of muds and lavas [11].

The basic assumptions of lubrication theory are directly applicable to several thixotropic flows of industrial or scientific interest, such as the motion of a thin layer of mud on a slope [12,13] or the flow of drilling muds or waxy crude oils in pipelines [14]. Lubrication scalings of the governing equations have been employed in several studies [15–18] to simplify the governing equations before integrating them numerically. Lubrication theory may also provide a useful starting point for investigating thixotropy in other contexts, such as porous media, where, despite a need which was identified over a decade ago by Pearson and Tardy [19], satisfactory models of thixotropic flow have yet to emerge.

Although no general theory of lubrication flow has hitherto been developed for thixotropic fluids, several recent studies have presented models which help to point the way to such a theory.

* Corresponding author.
E-mail addresses: david.pritchard@strath.ac.uk (D. Pritchard), s.k.wilson@strath.ac.uk (S.K. Wilson).

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In the flow realised, for example, in cone-and-plate rheometers, the shear rate is uniform in both the streamwise and the transverse directions. Thus, even for a thixotropic fluid, the relationship between shear stress and shear rate can be described in terms of ordinary differential equations. Several studies [20–22] have extended this to configurations in which the shear rate may vary in the transverse direction but remains uniform in the streamwise direction: this represents a limiting case of the lubrication regime.

In a preliminary study of such a configuration, Coussot et al. [12] modelled the acceleration of a uniform layer of fluid on an inclined plane in terms of a layer-averaged streamwise velocity and a layer-averaged structure parameter. This further reduction of the equations recovers the simplicity of a purely time-dependent problem, at the cost of the *ad hoc* assumption that the dynamics are well represented by layer-averaged quantities.

Similar *ad hoc* reductions have been employed to model flows that were evolving both in the streamwise direction and in time: Chanson et al. [23] considered dam-break flow on an inclined plane, while Pritchard and Pearson [24] considered flow in a narrow fracture, taken to be equivalent to Darcy flow in a porous medium. Both studies reduced the governing equations on the assumption that the rheological state of the fluid in a given cross-section could be characterised by a single quantity: [23] employed a "vertically averaged" value of the structure parameter, while [24] employed a "cross-sectionally averaged" value of the fluidity in a version of Bautista et al.'s [25] model.

The study by Livescu et al. [26], who considered the levelling of a thin film of thixotropic fluid on a horizontal substrate, represents a bridge between lubrication theory and reduced models. They simplified the governing hydrodynamic equations using a lubrication approximation, then integrated them numerically, and proposed a reduced model based on these numerical results. This approach is an advance on that of [23] and [24], because it does not postulate in advance that the transverse variation of the structure is known. However, the weakness of this approach is that the transverse variation must be obtained by numerical simulations of a non-reduced system, and there is no guarantee that the approximate profiles for λ obtained in this way will be equally applicable to different rheologies or to different problems.

With this in mind, our goal in this paper is to systematically develop the governing equations for lubrication flow of thixotropic and antithixotropic fluids in a slowly varying geometry. Given the uncertainties involved in the rheological characterisation of thixotropic fluids [2,14], we will develop this lubrication theory as generally as possible, instead of following most previous studies by restricting our discussion to a specific rheology from the start.

One category of behaviour exhibited by structure-parameter models will not be discussed here, although our approach could in principle be extended to include it. For certain choices of the kinetic model that determines the evolution of λ, even in steady uniform flow λ may have multiple equilibrium values for a given shear rate [20,27]. This non-uniqueness in turn causes non-monotonicity in the equilibrium stress–strain-rate curve and non-uniqueness of the equilibrium flow profiles. When the structure response time is very short, this behaviour may be described by considering a non-unique stress–strain-rate relation and tracking which branch of this relation applies at each point in the flow. If local flow conditions alter so that a solution on a given branch is no longer available, a "viscosity bifurcation" occurs and the structure is assumed to adjust immediately to another branch. (Here we use the term "viscosity bifurcation" in the sense of Hewitt and Balmforth [27], who incorporated this behaviour in a model of thin-film flow and tracked the surfaces in the flow where viscosity bifurcations occured.) While non-uniqueness is certainly worthy of further study and may be associated with important physical phenomena such as shear banding [28,29], we do not regard it as the defining feature of thixotropic flow and so will not discuss it here. Moreover, ruling out non-uniqueness allows us to formulate our leading-order solutions in a convenient analytical form. For similar technical reasons we will not consider true yield-stress behaviour, although we will consider the behaviour of a regularised yield-stress model. We note that although in some materials thixotropy and yield stress are intimately linked phenomena, each may occur without the other [2,30], so this is also not a fundamental restriction on the present analysis.

In Section 2 we present the governing equations for thixotropic and antithixotropic fluids, and a systematic expansion of these equations for lubrication flow. In the course of this derivation we define an advective Deborah number $\mathcal{D}$, and we show that different regimes may be identified in terms of the relative magnitudes of this Deborah number and the small aspect ratio $\delta \ll 1$ employed in the lubrication expansion. In Section 3 we specialise to the "weakly advective" regime $\mathcal{D} = \mathcal{O}(\delta)$, and develop semi-analytical solutions for general constitutive laws and structure evolution laws. In Section 4 we present illustrative results for two thixotropic models: the purely viscous Moore–Mewis–Wagner model and a regularised version of the viscoplastic Houška model. In particular, we discuss the flow profiles across the channel, and consider pressure gradients and pressure drops in channels of specified shape. In Section 5 we investigate the behaviour of a reduced Darcy model for channel flow, and show how lubrication theory can be used both to calibrate such models and to assess their validity. Finally, in Section 6 we summarise our results and discuss directions for the further development of our approach.

2. Derivation of the lubrication equations

2.1. Governing equations and boundary conditions

We consider steady, two-dimensional flow of an incompressible thixotropic or antithixotropic fluid at zero Reynolds number. This flow is governed by the continuity equation

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \tag{1}$$

where $\hat{u}(\hat{x},\hat{y})$ and $\hat{v}(\hat{x},\hat{y})$ are the velocity components in the $\hat{x}$ and $\hat{y}$ directions respectively, together with the steady generalised Cauchy momentum equations

$$\frac{\partial \hat{p}}{\partial \hat{x}} = \frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{xy}}{\partial \hat{y}} \quad \text{and} \quad \frac{\partial \hat{p}}{\partial \hat{y}} = \frac{\partial \hat{\tau}_{yx}}{\partial \hat{x}} + \frac{\partial \hat{\tau}_{yy}}{\partial \hat{y}}, \tag{2}$$

where $\hat{p}(\hat{x},\hat{y})$ is the pressure, and where the shear stress tensor $\hat{\tau}_{ij}$ depends on the shear rate tensor $\hat{e}_{ij}$ and on the structure parameter $\lambda(\hat{x},\hat{y})$. Here and elsewhere a caret denotes a dimensional quantity while dimensionless quantities are unadorned.

More specifically, we consider an ideal thixotropic fluid (in the sense of Larson [31]) and take the shear stress tensor to be of generalised Newtonian form,

$$\hat{\tau}_{ij} = \hat{\eta}(\dot{\gamma},\lambda)\hat{e}_{ij}, \tag{3}$$

for an apparent viscosity $\hat{\eta}$ that depends on both the total shear rate $\dot{\gamma}$ and on the local state of the microstructure represented by λ. The momentum equations (2) thus become

$$\frac{\partial \hat{p}}{\partial \hat{x}} = \frac{\partial}{\partial \hat{x}}\left[2\hat{\eta}\frac{\partial \hat{u}}{\partial \hat{x}}\right] + \frac{\partial}{\partial \hat{y}}\left[\hat{\eta}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right)\right] \tag{4}$$

and

$$\frac{\partial \hat{p}}{\partial \hat{y}} = \frac{\partial}{\partial \hat{x}}\left[\hat{\eta}\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right)\right] + \frac{\partial}{\partial \hat{y}}\left[2\hat{\eta}\frac{\partial \hat{v}}{\partial \hat{y}}\right]. \tag{5}$$

The steady evolution equation for the structure parameter must represent the advection of microstructure along with its build-up