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On the stability of the BEK family of rotating boundary-layer flows for power-law fluids

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ABSTRACT

We consider the convective instability of the BEK family of rotating boundary-layer flows for shear-thinning power-law fluids. The Bödewadt, Ekman and von Kármán flows are particular cases within this family. A linear stability analysis is conducted using a Chebyshev polynomial method in order to investigate the effect of shear-thinning fluids on the convective type I (inviscid crossflow) and type II (viscous streamline curvature) modes of instability. The results reveal that an increase in shear-thinning has a universal stabilising effect across the entire BEK family. Our results are presented in terms of neutral curves, growth rates and an analysis of the energy balance. The newly-derived governing equations for both the steady mean flow and unsteady perturbation equations are given in full.

1. Introduction

There has been significant interest in the stability and transition of the three-dimensional boundary-layer flow due to the rotating disk (that is the von Kármán [1] flow) in recent decades. The seminal study of the stability properties of the Newtonian rotating-disk boundary layer was performed by Gregory et al. [2], and there the first experimental observation of stationary crossflow vortices and the first theoretical stability analysis are presented. Some years later, Malik [3], utilsing the parallel-flow approximation, extended Gregory et al.'s high-Reynolds-number analytics and computed the neutral curves for stationary disturbances. Malik identified two distinct instability modes. The first mode (denoted type I), due to inviscid crossflow instability, was shown to be the dominant mode and was associated with Gregory et al.'s prior results. The additional second mode (denoted type II) was shown to be viscous in nature and attributed to external streamline curvature and Coriolis forces. In the same year Hall [4], approached the problem rigorously and presented a high-Reynolds-number linear asymptotic analysis. Complete agreement between Hall and Malik's studies is found in the appropriate parameter limit.

Following these important milestones the seemingly simple system has continued to attract attention and it remains under active investigation to this day. The interested reader is referred to the literature for full information on the latest developments concerning the transition to turbulence via the generation of a non-linear global mode (see, for example, [5–8]).

This paper is concerned with the stability characteristics of the family of boundary-layer flows attributed to a differential rotation rate between a lower disk and upper fluid in rigid-body rotation. Particular arrangements of this dual rotating system include the von Kármán [1], Ekman [9], and Bödewadt [10] boundary-layer flows. The von Kármán boundary layer arises when the lower disk rotates under a stationary fluid, the Ekman layer occurs when the disk and fluid rotate with approximately the same angular velocity, and the Bödewadt layer occurs when the fluid rotates above a stationary disk. There is a continuum of intermediate cases between these standard configurations and collectively these form the BEK family.

The mean-flow solutions of the entire BEK family for Newtonian flows are well understood [11–13]. In contrast, the stability characteristics of this family of flows has received relatively minimal attention, motivated largely by a desire to simply generalise the active research on the rotating-disk (i.e. von Kármán) system. In particular, Lingwood [14] presents local convective and absolute instability analyses of the Newtonian boundary layer and concludes that the limiting case of the rotating disk is the most stable configuration within the family. More recently, Lingwood and Garrett [15] discuss the use of mass flux through the lower disk as a po-

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tential flow-control mechanism. Various experimental studies concerning the stability, transition and control of these types of flows has been an area of more recent active research [16–18].

It is our intention here to generalise Lingwood's original work in this area to incorporate the effects of non-Newtonian fluids. Our motivation is to explore the potential for using such fluids to optimise the performance of rotor-stator systems in engineering applications.

With regards to prior studies of the non-Newtonian boundary-layer flow over a rotating disk, Mitschka and Ulbrecht [19] were the first to extend the von Kármán similarity solution to incorporate fluids that adhere to a power-law governing viscosity relationship. That study, involving both shear-thickening and shear-thinning fluids, was later verified by Andersson et al. [20] in order to test the reliability of their numerical solutions. However, further to this, Denier and Hewitt [21] readdressed the problem showing that asymptotic matching considerations need to be taken in to account in order to able to accurately describe the flow of shear-thinning power-law fluids. In the shear-thickening regime it transpires that the boundary-layer solutions are complicated by a region of zero viscosity away from the wall. For these reasons, in this study, we will restrict our attention to moderately shear-thinning fluids only. For full details regarding the asymptotic structure of the solutions the interested reader is refereed to Denier and Hewitt [21].

Much more recently Griffiths et al. [22] considered a rigorous asymptotic stability analysis of the shear-thinning boundary-layer flow over a rotating disk. This work was then extended by the same authors Griffiths et al. [23], to compute the neutral curves of convective instability (working under the parallel-flow assumption) and complete agreement was found with their prior asymptotic analysis. These two papers can be considered as the non-Newtonian generalisations of Hall [4] and Malik [3], respectively. Griffiths [24] later extends the power-law studies to include the Bingham [25] and Carreau [26] models of non-Newtonian viscosity. He finds that a generalisation of the von Kármán similarity solution is applicable for a variety of different inelastic and viscoplastic non-Newtonian models.

In this current paper we extend the non-Newtonian, inelastic study of Griffiths et al. [23], to the entire BEK family of rotating boundary-layer flows. A Chebyshev polynomial method is used to consider the effects shear-thinning power-law fluids have on the type I and type II modes of instability.

This paper proceeds as follows: In Section 2 the steady boundary-layer flows of the BEK system for fluids with a governing viscosity relationship adhering to a power-law model are formulated and the profiles presented. A local convective instability analysis is presented in Section 3 and the neutral curves, critical Reynolds numbers, and convective growth rates are detailed for a variety of flow parameters. An energy-balance analysis is considered in Section 4 and finally our conclusions are drawn in Section 5. All newly-derived equations are presented in detail where appropriate in our discussion.

2. Formulation

We consider a family of incompressible, shear-thinning boundary-layer flows above an infinite rotating disk located at $z^*=0$. Distinct flows within this family are generated by a differential rotation rate between this solid boundary and a fluid in rigid-body rotation (see [14,15]). Particular cases within the family are the Bödewadt, Ekman and von Kármán boundary layer flows and we denote the entire family as the *BEK system*. Both rotating components (disk and fluid) are assumed to rotate in the same direction and about the same vertical axis with angular velocities Ω_D^* and Ω_F^*, respectively. The von Kármán layer appears when the fluid is stationary and the disk rotates, i.e., $\Omega_F^*=0$ and $\Omega_D^*\neq 0$; the Ekman layer is such that $\Omega_F^*\approx\Omega_D^*$; and the Bödewadt is such that $\Omega_F^*\neq 0$ and $\Omega_D^*=0$. Furthermore, there exists a continuum of cases between these particular examples in which both the disk and fluid rotate with different angular velocities.

The continuity and Navier–Stokes equations in a frame of reference rotating with the lower disk, at fixed angular velocity, are expressed as follows

$$\nabla\cdot\boldsymbol{u}^*=0, \tag{1a}$$

$$\frac{\partial \boldsymbol{u}^*}{\partial t^*}+\boldsymbol{u}^*\cdot\nabla\boldsymbol{u}^*+\boldsymbol{\Omega}^*\times(\boldsymbol{\Omega}^*\times\boldsymbol{r}^*)+2\boldsymbol{\Omega}^*\times\boldsymbol{u}^* = -\frac{1}{\rho^*}\nabla p^*+\frac{1}{\rho^*}\nabla\cdot\boldsymbol{\tau}^*. \tag{1b}$$

Here $\boldsymbol{u}^*=(U^*,V^*,W^*)$ are the velocity components in cylindrical polar coordinates (r^*,θ,z^*), t^* is time, $\boldsymbol{\Omega}^*=(0,0,\Omega^*)$ and $\boldsymbol{r}^*=(r^*,0,z^*)$. The fluid density is ρ^* and p^* is the fluid pressure.

The stress tensor $\boldsymbol{\tau}^*$ for generalised Newtonian models, such as the power-law model, is defined by

$$\boldsymbol{\tau}^*=\mu^*\dot{\boldsymbol{\gamma}}^* \quad \text{with} \quad \mu^*=\mu^*(\dot{\gamma}^*),$$

where $\dot{\boldsymbol{\gamma}}^*=\nabla\boldsymbol{u}^*+(\nabla\boldsymbol{u}^*)^{\mathrm{T}}$ is the rate of strain tensor and $\mu^*(\dot{\gamma}^*)$ is the non-Newtonian viscosity. The magnitude of the rate of strain tensor is given by

$$\dot{\gamma}^*=\sqrt{\frac{\dot{\boldsymbol{\gamma}}^*:\dot{\boldsymbol{\gamma}}^*}{2}}.$$

For power-law fluids the governing relationship for $\mu^*(\dot{\gamma}^*)$ is

$$\mu^*(\dot{\gamma}^*)=m^*(\dot{\gamma}^*)^{n-1}, \tag{2}$$

where m^* is the consistency coefficient and n is the dimensionless power-law index. For $n<1$ we have a *pseudoplastic fluid* where the viscosity decreases with increased rate of strain (i.e., shear thinning). For $n>1$ we have a *dilitant fluid* where the viscosity increases with increased rate of strain (i.e., shear thickening). The classical Newtonian viscosity law is recovered for the particular parameter value $n=1$.

The governing boundary-layer equations are formulated in a frame rotating with the lower disk, i.e., at Ω_D^*, and are expressed in cylindrical-polar coordinates (r^*,θ,z^*) as

$$\frac{1}{r^*}\frac{\partial(r^*U_0^*)}{\partial r^*}+\frac{1}{r^*}\frac{\partial V_0^*}{\partial\theta}+\frac{\partial W_0^*}{\partial z^*}=0, \tag{3a}$$

$$\frac{\partial U_0^*}{\partial t^*}+U_0^*\frac{\partial U_0^*}{\partial r^*}+\frac{V_0^*}{r^*}\frac{\partial U_0^*}{\partial\theta}+W_0^*\frac{\partial U_0^*}{\partial z^*}-\frac{(V_0^*+r^*\Omega_D^*)^2}{r^*} = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu_0^*\frac{\partial U_0^*}{\partial z^*}\right), \tag{3b}$$

$$\frac{\partial V_0^*}{\partial t^*}+U_0^*\frac{\partial V_0^*}{\partial r^*}+\frac{V_0^*}{r^*}\frac{\partial V_0^*}{\partial\theta}+W_0^*\frac{\partial V_0^*}{\partial z^*}+\frac{U_0^*V_0^*}{r^*}+2\Omega_D^*U_0^* = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu_0^*\frac{\partial V_0^*}{\partial z^*}\right), \tag{3c}$$

$$\frac{\partial W_0^*}{\partial t^*}+U_0^*\frac{\partial W_0^*}{\partial r^*}+\frac{V_0^*}{r^*}\frac{\partial W_0^*}{\partial\theta}+W_0^*\frac{\partial W_0^*}{\partial z^*}=-\frac{1}{\rho^*}\frac{\partial P_1^*}{\partial z^*} +\frac{1}{\rho^*r^*}\frac{\partial}{\partial r^*}\left(\mu_0^*r^*\frac{\partial U_0^*}{\partial z^*}\right)+\frac{1}{\rho^*r^*}\frac{\partial}{\partial\theta}\left(\mu_0^*\frac{\partial V_0^*}{\partial z^*}\right) +\frac{2}{\rho^*}\frac{\partial}{\partial z^*}\left(\mu_0^*\frac{\partial W_0^*}{\partial z^*}\right), \tag{3d}$$