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# A comparison of implicit solvers for the immersed boundary equations

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**Abstract**

Explicit time discretizations of the immersed boundary method are known to require small timesteps to maintain stability. A number of implicit methods have been introduced to alleviate this restriction to allow for a more efficient method, but many of these methods still have a stability restriction on the timestep. Furthermore, almost no comparisons have appeared in the literature of the relative computational costs of the implicit methods and the explicit method. A recent paper [E.P. Newren, A.L. Fogelson, R.D. Guy, R.M. Kirby, Unconditionally stable discretizations of the immersed boundary equations, J. Comput. Phys. 222 (2007) 702–719.] addressed the confusion over stability of immersed boundary discretizations. This paper identified the cause of instability in previous immersed boundary discretizations as lack of conservation of energy and introduced a new semi-implicit discretization proven to be unconditionally stable, *i.e., it has bounded discrete energy*. The current paper addresses the issue of the efficiency of the implicit solvers. Existing and new methods to solve implicit immersed boundary equations are described. Systematic comparisons of computational cost are presented for a number of these solution methods for our stable semi-implicit immersed boundary discretization and an explicit discretization for two distinct test problems. These comparisons show that two of the implicit methods are at least competitive with the explicit method on one test problem and outperform it on the other test problem in which the elastic stiffness of the boundary does not dictate the timescale of the fluid motion.


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## 1. Introduction

The immersed boundary (IB) method was introduced by Peskin in the early 1970s to solve the coupled equations of motion of a viscous, incompressible fluid and one or more massless, elastic surfaces or objects immersed in the fluid [24]. Rather than generating a curve-fitting grid for both exterior and interior regions of each surface at each timestep and using these to determine the fluid motion, Peskin instead employed a uniform Cartesian grid over the entire domain and discretized the immersed boundaries by a set of points that are *not* constrained to lie on the grid. The key idea that permits this simplified discretization is the replacement of each suspended object by a suitable contribution to a force density term in the fluid dynamics equations in order to allow those equations to hold in the entire domain with no internal boundary conditions.

The IB method was originally developed to model blood flow in the heart and through heart valves [24,26,27], but

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has since been used in a wide variety of other applications, particularly in biofluid dynamics problems where complex geometries and immersed elastic membranes or structures are present and make traditional computational approaches difficult. Examples include platelet aggregation in blood clotting [9,11], swimming of micro-organisms [9,10], biofilm processes [8], mechanical properties of cells [1], cochlear dynamics [3], and insect flight [18,19]. We refer the reader to [25] for a more extensive list of applications.

The immersed interface (II) method was developed by Leveque and Li to address the low order accuracy found in the IB method when applied to problems with sharp interfaces [16]. The II method differs from the IB method in the spatial discretization method of the singular forces appearing in the continuous equations of motion. While we do not address the spatial discretizations involved in the II method and instead focus on the IB method in this paper, we do present some discussion of that method since the two are closely related, in fact hybrids of the two exist, such as [15].

Explicit timestepping with the IB and II methods leads to a severe timestep restriction in order to maintain stability [9,16,25,29]. This time step restriction is typically much more stringent than one that would be imposed if explicit differencing of the advective or diffusive terms [6] were used. Much effort has been expended attempting to alleviate this severe restriction, including the development of various implicit and semi-implicit methods [4,9,16,17,20,30,31].

The use of implicit methods for solving the IB equations has met with very limited success. Until the recent results in [22], no implicit IB methods were known to be unconditionally stable, and the observed instability even of some of the fully implicit methods was not well understood. There has also been an almost complete lack of computational comparisons of implicit methods with the explicit method. In fact, despite the many papers introducing implicit methods for solving the IB equations, very few of them have done any concrete comparisons of the computational cost of their implicit methods to the explicit method; two of these papers [22,30] have stated that their implicit method was slower than the explicit method, while others have simply overlooked comparing their implicit method with the explicit method in terms of CPU time. The only works of which we know to concretely compare computational cost were that of Stockie and Wetton [29], whose main focus was an analysis of IB stability, and the work of Mori and Peskin [20] found in this issue.

The issue of stability of implicit discretizations of the IB equations was addressed in [22], where the authors showed that previously suspected and asserted causes of numerical instability for the IB method were not the actual sources of instability and identified the cause of instability in previous implicit IB discretizations as a lack of conservation of energy of the numerical discretization. In [22], a new semi-implicit discretization which was proven to be unconditionally stable in the sense that a natural discrete energy was bounded. An intriguing consequence of that work is that linear solvers can be used on stable IB equations. Prior to the results of [22], it was commonly believed that only fully implicit discretizations could produce an unconditionally stable immersed boundary method. Fully implicit discretizations lead to systems of equations that are nonlinear in the IB point locations, and this nonlinearity reduces the range of applicable solvers.

In this paper, we seek to address the efficiency of implicit solvers for the IB method. In particular, we look at methods that take advantage of the ability to use linear solvers afforded to us by the new stable semi-implicit discretization of [22]. Since many implicit solvers have already been introduced in the literature, we begin by cataloguing these methods and discussing their effectiveness and applicability. We also introduce several new methods that exploit the linearity of our stable implicit equations, and then compare their computational cost with that of an explicit method.

In Section 2, we review the immersed boundary equations of motion and their stable discretization. In Section 3, we catalog and discuss existing and new approaches to solving implicit IB equations, and in Section 4, we present detailed comparisons of the relative computational efficiency of some of these implicit solvers and the explicit method.

## 2. The immersed boundary method

In the IB method, an Eulerian description is used for the fluid variables, and a Lagrangian description is used for each object immersed in the fluid. The boundary is assumed to be massless, so that all of the force generated by distortions of the boundary is transmitted directly to the fluid. An example setup in 2D with a single immersed boundary curve is shown in Fig. 1. Lowercase letters are used for Eulerian state variables, while uppercase letters are used for Lagrangian variables. Thus, $\mathbf{X}(s,t)$ is a vector

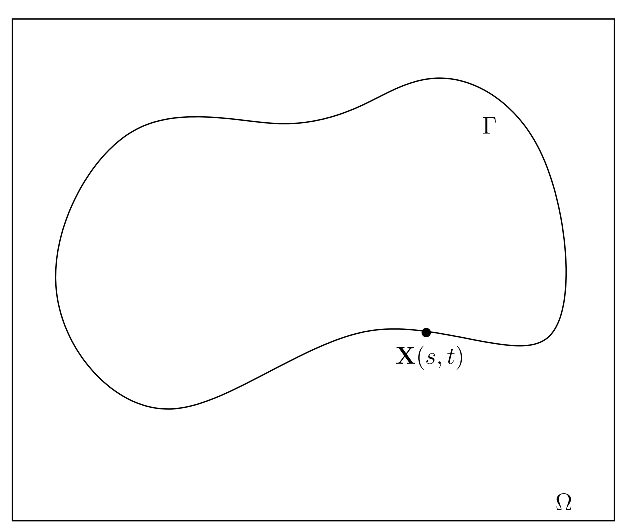


Fig. 1. Example immersed boundary curve, $\Gamma$, described by the function $\mathbf{X}(s,t)$, immersed in a fluid-filled region $\mathbf{\Omega}$.