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hp-version interior penalty DGFEMs for the biharmonic equation ☆

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Abstract

We construct *hp*-version interior penalty discontinuous Galerkin finite element methods (DGFEMs) for the biharmonic equation, including symmetric and nonsymmetric interior penalty discontinuous Galerkin methods and their combinations: semisymmetric methods. Our main concern is to establish the stability and to develop the *a priori* error analysis of these methods. We establish error bounds that are optimal in h and slightly suboptimal in p. The theoretical results are confirmed by numerical experiments.

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1. Introduction

Conforming finite element methods for the numerical solution of boundary value problems for the biharmonic equation require that the approximate solution lie in a finite-dimensional subspace of the Sobolev space $H^2(\Omega)$. In particular, this necessitates the use of C^1 finite elements; i.e., the basis functions of the finite element space, together with their first partial derivatives, need to be continuous over $\overline{\Omega}$. Because the construction of such finite element spaces is fairly involved, $H^2(\Omega)$-conforming finite elements are rarely used in practical computations. One way to relax these regularity requirements is to use nonconforming methods which rely on continuous finite element basis functions that do not belong to $C^1(\overline{\Omega})$ (and are, therefore, not included in $H^2(\Omega)$ either). For details, see [13,14] and references therein.

Other approaches to avoid the use of C^1 finite elements include hybrid and mixed finite element methods. The literature on mixed methods is extensive, and we refer to the survey paper [31] and the monograph of Brezzi and Fortin [11] for general results concerning the construction and the analysis of these methods. Some applications of mixed and hybrid methods for the biharmonic problem are presented in [21,34,19,12].

In recent years discontinuous Galerkin finite element methods (DGFEMs) have been widely used for the numerical solution of a large range of computational problems for partial differential equations, including linear and nonlinear hyperbolic problems, convection-dominated diffusion problems and second-order elliptic problems. For an excellent historical survey of the subject, a summary of various DGFEMs, and an extensive list of references, we refer to the special volume [15].

Already in some of the early papers on DGFEMs for second-order elliptic equations the bilinear form of the method included interior penalty terms to penalize jumps across element faces in the numerical solution (cf. [23,37,2]) and to ensure the coercivity of the bilinear form. More recently, starting from the same basic premise, several authors introduced DGFEMs for second-order elliptic problems, such as the nonsymmetric interior penalty DGFEM (NIPG) (cf. [28–30,36,18]), and the symmetric

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interior penalty DGFEM (SIPG) (cf. [2,37]). A discontinuous Galerkin finite element method without interior penalty terms was proposed by Baumann and Oden [8,24]. A detailed study and a unified error analysis of DGFEMs for second-order elliptic problems is given in the paper by Arnold at al. [3].

An interior penalty finite element method for fourth-order elliptic equations was proposed in [16] and in [6]. More recently this approach was further developed in [17], where a continuous/discontinuous Galerkin method, which combines concepts from the theory of continuous and discontinuous Galerkin methods with ideas from the theory of stabilized methods, was proposed for fourth-order elliptic partial differential equations. The methods developed in that work were based on the symmetric version of the Interior Penalty Galerkin method; the Nonsymmetric Interior Penalty Galerkin Method for fourth-order elliptic equations was considered in [22].

The purpose of this paper is to extend to higher-order elliptic equations the *hp*-version of the interior penalty DGFEM in symmetric and nonsymmetric formulations. Our main concern is to establish the stability of these methods and to derive *a priori* error bounds. For reasons of clarity of exposition, we consider the simple case of the Dirichlet problem for the biharmonic equation, although the basic ideas developed here are readily extendable to linear elliptic operators of order $2m$ for any $m \geqslant 1$.

The paper is structured as follows. In Section 2, we introduce finite element spaces consisting of discontinuous piecewise polynomials and broken Sobolev spaces. Then we formulate, in Section 3, the model boundary value problem for the biharmonic equation; we consider the broken weak formulation of the problem, show the consistency of this formulation leading to a Galerkin orthogonality property, and demonstrate the boundedness of the associated bilinear form. In Section 4 we present a family of Discontinuous Galerkin methods for the biharmonic equation, which includes NIPG and SIPG methods and their combinations: the semisymmetric methods SSIPG1 and SSIPG2. In this section, we also prove the coercivity of the general bilinear form and deduce from this result the coercivity of these four methods, for suitable choices of the penalty parameters. In Section 5 we prove *hp*-version *a priori* error bounds in the energy norm for the interior penalty Galerkin methods introduced in Section 4. First, using the coercivity results from Section 4, we prove *hp*-version error bounds for each of NIPG, SIPG, SSIPG1 and SSIPG2, following the ideas from [26,27]. Then, in the case of the NIPG method, we present an alternative *hp*-version error analysis, inspired by the results of [18]; thus we obtain the same order of convergence but with a weaker restriction on the size of the penalty parameters with respect to the polynomial degree p (cf. [22]). In particular, we establish error bounds that are optimal in h and suboptimal in p. Section 6 presents a series of numerical experiments which confirm the theoretically predicted convergence rates.

2. Finite element spaces

Suppose that Ω is a bounded, open, convex polyhedral domain in $\mathbb{R}^d$, $d \geqslant 2$, with boundary $\partial\Omega$ which is the union of its open $(d-1)$-dimensional faces. Let us consider a family of triangulations $\mathscr{K}_h$ of Ω, parametrized by $h > 0$. That is, for each $h > 0$, $\mathscr{K}_h$ is a partition of Ω into disjoint open convex polyhedral element domains $K = K_j$ such that $\overline{\Omega} = \bigcup_{K\in\mathscr{K}_h}\overline{K}$, $K_i \cap K_j = \emptyset$ for $i \neq j$ and the intersection $\overline{K}_i \bigcap \overline{K}_j$ is either empty, a vertex, an edge or a face. We define a piecewise constant mesh function $h_{\mathscr{K}}$ by

$$h_{\mathscr{K}}(x) = h_K = \operatorname{diam}(K), \quad x \in K, \ K \in \mathscr{K}$$

and put

$$h = \max_{K\in\mathscr{K}_h} h_K.$$

Let $\widehat{K}$ be a fixed master element in $\mathbb{R}^d$; here we shall suppose that $\widehat{K}$ is the open unit hypercube in $\mathbb{R}^d$. We shall further assume that each $K \in \mathscr{K}_h$ is an affine image of the master element $\widehat{K}$:

$$K = F_K(\widehat{K}), \quad K \in \mathscr{K}_h.$$

Let $\mathscr{E}$ be the set of all open $(d-1)$-dimensional faces of all elements $K \in \mathscr{K}_h$. We also define a piecewise constant face-function on $\mathscr{E}$:

$$h_{\mathscr{E}}(x) = h_e = \operatorname{diam}(e), \quad x \in e, \ e \in \mathscr{E}.$$

Let us assume that the family of triangulations $\{\mathscr{K}_h\}_{h>0}$ is shape-regular (cf. Remark 2.2, p. 114, in [10]). We note that for a shape-regular family there exists a positive constant c (the shape-regularity constant), independent of h, such that

$$ch_K \leqslant h_e \leqslant h_K, \quad \forall K \in \bigcup_{h>0} \mathscr{K}_h \ \forall e \in \partial K, \tag{1}$$

hence, for any element $K \in \mathscr{K}_h$, h_K and h_e are equal to within a constant.

For a nonnegative integer m, we denote by $\mathscr{Q}_m(\widehat{K})$ the set of all tensor-product polynomials of degree m or less in each coordinate direction. Then, to each $K \in \mathscr{K}_h$ we assign a nonnegative integer p_K (the local polynomial degree) and a nonnegative integer s_K (the local Sobolev space index). Collecting the p_K, s_K and F_K in the vectors $\mathbf{p} = (p_K : K \in \mathscr{K}_h)$, $\mathbf{s} = (s_K : K \in \mathscr{K}_h)$ and $\mathbf{F} = (F_K : K \in \mathscr{K}_h)$, respectively, we introduce the finite element space

$$S^{\mathbf{p}}(\Omega, \mathscr{K}_h, \mathbf{F}) = \{u \in L^2(\Omega) : u|_K \circ F_K \in \mathscr{Q}_{p_K}(\widehat{K}) \quad \forall K \in \mathscr{K}_h\}.$$

Moreover we define, for the triangulation $\mathscr{K}_h$, the broken Sobolev space

$$H^{\mathbf{s}}(\Omega, \mathscr{K}_h) = \{u \in L^2(\Omega) : u|_K \in H^{s_K}(K) \ \forall K \in \mathscr{K}_h\}$$

equipped with the broken Sobolev norm and seminorm

$$\|u\|_{\mathbf{s},\mathscr{K}_h} = \left(\sum_{K\in\mathscr{K}_h} \|u\|^2_{H^{s_K}(K)}\right)^{\frac{1}{2}}, \quad |u|_{\mathbf{s},\mathscr{K}_h} = \left(\sum_{K\in\mathscr{K}_h} |u|^2_{H^{s_K}(K)}\right)^{\frac{1}{2}}.$$