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The Coulomb gauged vector potential formulation for the eddy-current problem in general geometry: Well-posedness and numerical approximation

Oszkár Bíró [a], Alberto Valli [b,*]

[a] Institute for Fundamentals and Theory in Electrical Engineering, Graz University of Technology, Kopernikusgasse 24, A-8010 Graz, Austria
[b] Dipartimento di Matematica, Università di Trento, 38050 Povo (Trento), Italy

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Abstract

In this paper we prove that, in a general geometric situation, the Coulomb gauged vector potential formulation of the eddy-current problem for the time-harmonic Maxwell equations is well-posed, i.e., its solution exists and is unique. Moreover, a quasi-optimal error estimate for its finite element approximation with nodal elements is proved. To illustrate the performances of the finite element algorithm, some numerical results are also presented.

Keywords: Eddy-currents; Vector potentials; Coulomb gauge; Well-posedness; Finite element approximation

1. Introduction

Let us consider a bounded connected open set $\Omega \subset \mathbf{R}^3$, with boundary $\partial\Omega$. The unit outward normal vector on $\partial\Omega$ will be denoted by $\mathbf{n}$. We assume that $\overline{\Omega}$ is split into two parts, $\overline{\Omega} = \overline{\Omega_C} \cup \overline{\Omega_I}$, where Ω_C (a non-homogeneous non-isotropic conductor) and Ω_I (a perfect insulator) are open disjoint subsets, such that $\overline{\Omega_C} \subset \Omega$. For the sake of simplicity, we also suppose that Ω_I is connected (the general case can be treated in a similar way, focusing on each connected component of Ω_I, but some modifications are needed when the boundary of a connected component of Ω_I has empty intersection with $\partial\Omega$).

We denote by $\Gamma := \partial\Omega_I \cap \partial\Omega_C$ the interface between the two subdomains; note that, in the present situation, $\partial\Omega_C = \Gamma$ and $\partial\Omega_I = \partial\Omega \cup \Gamma$. Moreover, we indicate by Γ_j, $j = 1,\dots,p_\Gamma$, the connected components of Γ, and by $(\partial\Omega)_t$, $t = 0,1,\dots,p_{\partial\Omega}$, the connected components of $\partial\Omega$ (in particular, we have denoted by $(\partial\Omega)_0$ the external one).

In this paper we study the time-harmonic *eddy-current problem*, in which the displacement current term $\frac{\partial\mathscr{D}}{\partial t}$ is neglected, and the electric field $\mathscr{E}$, the magnetic field $\mathscr{H}$ and the applied current density $\mathscr{J}_e$ are of the form

$$\mathscr{E}(t,\mathbf{x}) = \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(\mathrm{i}\omega t)],$$
$$\mathscr{H}(t,\mathbf{x}) = \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(\mathrm{i}\omega t)],$$
$$\mathscr{J}_e(t,\mathbf{x}) = \mathrm{Re}[\mathbf{J}_e(\mathbf{x})\exp(\mathrm{i}\omega t)],$$

where $\omega \neq 0$ is a given angular frequency (see, e.g., [8, p. 219]).

The constitutive relation $\mathscr{B} = \mu\mathscr{H}$ (where μ is the magnetic permeability coefficient) is assumed to hold, as well as the (generalized) Ohm's law $\mathscr{J} = \sigma\mathscr{E} + \mathscr{J}_e$ (where σ is the electric conductivity).

The magnetic permeability μ is assumed to be a symmetric matrix, uniformly positive definite in Ω, with entries in $L^\infty(\Omega)$. Since Ω_I is a perfect insulator, we require that $\sigma_{|\Omega_I} \equiv 0$; moreover, as Ω_C is a non-homogeneous non-isotropic conductor, $\sigma_{|\Omega_C}$ is assumed to be a symmetric matrix,

* Corresponding author. Tel.: +39 0461 881580; fax: +39 0461 881624.
E-mail addresses: biro@igte.tu-graz.ac.at (O. Bíró), valli@science.unitn.it (A. Valli).

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uniformly positive definite in Ω_C, with entries in $L^\infty(\Omega_C)$. Moreover, the dielectric coefficient ε_I, which enters the problem when one has to determine the electric field in Ω_I, is assumed to be a symmetric matrix, uniformly positive definite in Ω_I, with entries in $L^\infty(\Omega_I)$. Finally, the applied current density $\mathscr{J}_e$ is not assumed to vanish in Ω_C, so that also the skin effect in current driven massive conductors can be modelled.

Concerning the boundary condition, we will consider in the *magnetic* boundary value problem, modelling a cavity within an infinitely permeable iron: namely, $\mathbf{H}\times\mathbf{n}$, representing the tangential component of the magnetic field, is assumed to vanish on $\partial\Omega$. The case of the *electric* boundary value problem, in which $\mathbf{E}\times\mathbf{n}$, representing the tangential component of the electric field, is assumed to vanish on $\partial\Omega$, can be treated following a similar approach, but in the sequel we will not dwell on it (its general description can be found, e.g., in [4]).

We will make the following assumptions on the geometry of Ω:

$$\begin{aligned}&\text{either } \partial\Omega\in C^{1,1}, \text{ or else } \Omega \text{ is a Lipschitz polyhedron;}\\ &\text{the same assumption holds for } \Omega_C \text{ and } \Omega_I.\end{aligned} \tag{H1}$$

As a consequence, it can be proved (see e.g., [16,21,5,15,18]) that the space of harmonic vector fields

$$\begin{aligned}\mathscr{H}_{\mu_I}(\partial\Omega;\Gamma) := \{\mathbf{v}_I\in(L^2(\Omega_I))^3 \,|\, \mathrm{curl}\,\mathbf{v}_I=\mathbf{0},\ \mathrm{div}(\mu_I\mathbf{v}_I)=0,\\ \mathbf{v}_I\times\mathbf{n}=\mathbf{0} \text{ on } \partial\Omega, \mu_I\mathbf{v}_I\cdot\mathbf{n}_I=0 \text{ on } \Gamma\}\end{aligned}$$

has a finite dimension, larger than $p_{\partial\Omega}$, say $p_{\partial\Omega}+n_\Gamma$, and that the space of harmonic vector fields

$$\begin{aligned}\mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega) := \{\mathbf{v}_I\in(L^2(\Omega_I))^3 \,|\, \mathrm{curl}\,\mathbf{v}_I=\mathbf{0}, \mathrm{div}(\varepsilon_I\mathbf{v}_I)=0,\\ \mathbf{v}_I\times\mathbf{n}_I=\mathbf{0} \text{ on } \Gamma, \varepsilon_I\mathbf{v}_I\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}\end{aligned}$$

has a finite dimension, larger than $p_\Gamma-1$, say $p_\Gamma-1+n_{\partial\Omega}$. Moreover

$$\begin{aligned}&\text{there exist } n_\Gamma \text{ "cuts" } \Xi_l, \text{ which are the interior of}\\ &\text{two-dimensional, mutually disjoint, compact and}\\ &\text{connected Lipschitz manifolds } \overline{\Xi_l} \text{ with boundary } \partial\Xi_l,\\ &\text{such that } \Xi_l\subset\Omega_I \text{ and } \partial\Xi_l\subset\Gamma, \text{ and such that in the}\\ &\text{open set } \widehat{\Omega_I}:=\Omega_I\setminus\cup_l\Xi_l, \text{ assumed to be connected,}\\ &\text{every curl-free vector field with vanishing tangential}\\ &\text{component on } \partial\Omega \text{ has a global potential;}\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\text{there exist } n_{\partial\Omega} \text{ "cuts" } \Sigma_k, \text{ which are the interior of}\\ &\text{two-dimensional, mutually disjoint, compact and}\\ &\text{connected Lipschitz manifolds } \overline{\Sigma_k} \text{ with boundary } \partial\Sigma_k,\\ &\text{such that } \Sigma_k\subset\Omega_I \text{ and } \partial\Sigma_k\subset\partial\Omega, \text{ and such that in the}\\ &\text{open set } \widetilde{\Omega_I}:=\Omega_I\setminus\cup_k\Sigma_k, \text{ assumed to be connected,}\\ &\text{every curl-free vector field with vanishing tangential}\\ &\text{component on } \Gamma \text{ has a global potential.}\end{aligned} \tag{1.2}$$

In particular, n_Γ is the number of *singular* (or *non-bounding*) cycles in $\overline{\Omega_I}$ of first type, namely those cycles on Γ that cannot be represented as $\partial S\backslash\gamma$, S being a surface contained in Ω_I and γ a cycle, possibly empty, contained in $\partial\Omega$. Similarly, $n_{\partial\Omega}$ is the number of singular cycles in $\overline{\Omega_I}$ of second type, namely, those cycles on $\partial\Omega$ that cannot be represented as $\partial S\backslash\gamma$, S being a surface contained in Ω_I and γ a cycle, possibly empty, contained in Γ.

Let us assume that the current density $\mathbf{J}_e\in(L^2(\Omega))^3$ satisfies the (necessary) conditions

$$\begin{aligned}&\mathrm{div}\,\mathbf{J}_{e,I}=0 \text{ in } \Omega_I,\quad \mathbf{J}_{e,I}\cdot\mathbf{n}=0 \text{ on } \partial\Omega,\\ &\int_{\Gamma_j}\mathbf{J}_{e,I}\cdot\mathbf{n}_I=0\quad\forall j=1,\dots,p_\Gamma-1,\\ &\int_{\Omega_I}\mathbf{J}_{e,I}\cdot\boldsymbol{\pi}_{k,I}=0\quad\forall k=1,\dots,n_{\partial\Omega},\end{aligned} \tag{H2}$$

where $\boldsymbol{\pi}_{k,I}$ are basis functions of the space $\mathscr{H}_{\varepsilon_I}(\Gamma;\partial\Omega)$ that are not gradients. As it will be shown in Section 4, condition $(\mathrm{H2})_3$ is equivalent to setting the total applied current through Σ_k to zero. This is necessary in view of Ampère law, since $\mathbf{H}\times\mathbf{n}$ vanishes on $\partial\Omega$, hence on $\partial\Sigma_k$.

(For the ease of the reader, in (H2) and in the sequel we are always denoting the duality pairings as surface integrals; see [8,13,17] for more details on these aspects related to functional analysis and to linear spaces of functions.)

In [4] it has been proved that the complete system of equations describing the eddy-current problem in terms of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}_C$ is:

$$\begin{cases}\mathrm{curl}\,\mathbf{E}_C+\mathrm{i}\omega\mu_C\mathbf{H}_C=\mathbf{0} & \text{in } \Omega_C,\\ \mathrm{curl}\,\mathbf{H}_C-\sigma\mathbf{E}_C=\mathbf{J}_{e,C} & \text{in } \Omega_C,\\ \mathrm{curl}\,\mathbf{H}_I=\mathbf{J}_{e,I} & \text{in } \Omega_I,\\ \mathrm{div}(\mu_I\mathbf{H}_I)=0 & \text{in } \Omega_I,\\ \int_{(\partial\Omega)_t}\mu_I\mathbf{H}_I\cdot\mathbf{n}=0 & \forall t=1,\dots,p_{\partial\Omega},\\ \int_{\Omega_I}\mathrm{i}\omega\mu_I\mathbf{H}_I\cdot\boldsymbol{\rho}_{l,I}+\int_\Gamma(\mathbf{E}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_{l,I}=0 & \forall l=1,\dots,n_\Gamma,\\ \mathbf{H}_I\times\mathbf{n}=\mathbf{0} & \text{on } \partial\Omega,\\ \mu_I\mathbf{H}_I\cdot\mathbf{n}_I+\mu_C\mathbf{H}_C\cdot\mathbf{n}_C=0 & \text{on } \Gamma,\\ \mathbf{H}_I\times\mathbf{n}_I+\mathbf{H}_C\times\mathbf{n}_C=\mathbf{0} & \text{on } \Gamma,\end{cases} \tag{1.3}$$

where $\mathbf{n}_C=-\mathbf{n}_I$ is the unit outward normal vector on $\partial\Omega_C=\Gamma$, we have set $\mathbf{E}_C:=\mathbf{E}_{|\Omega_C}$ (and similarly for Ω_I and any other restriction of function), and $\boldsymbol{\rho}_{l,I}$ are the basis functions of the space of harmonic fields $\mathscr{H}_{\mu_I}(\partial\Omega;\Gamma)$ that are not gradients. The non-local condition $(1.3)_6$ is needed to ensure that the Faraday law is satisfied on the cutting surfaces Ξ_l (see [4,3]). In particular, in [4] it has been proved that, under the assumptions (H1)–(H2), problem (1.3) has a unique solution.

In the following, we are going to consider problem (1.3) in terms of the Coulomb gauged vector potential formulation. We will prove that this formulation is well-posed,