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Domain decomposition methods for advection dominated linear-quadratic elliptic optimal control problems

Roscoe A. Bartlett [a,1], Matthias Heinkenschloss [b,*], Denis Ridzal [b], Bart G. van Bloemen Waanders [a]

[a] Optimization/Uncertainty Est Department (9211), Sandia National Laboratories, MS 0370, P.O. Box 5800, Albuquerque, NM 87185-0370, United States

[b] Department of Computational and Applied Mathematics, MS-134, Rice University, 6100 Main Street, Houston, TX 77005-1892, United States

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Abstract

We present an optimization-level domain decomposition (DD) preconditioner for the solution of advection dominated elliptic linear-quadratic optimal control problems, which arise in many science and engineering applications. The DD preconditioner is based on a decomposition of the optimality conditions for the elliptic linear-quadratic optimal control problem into smaller subdomain optimality conditions with Dirichlet boundary conditions for the states and the adjoints on the subdomain interfaces. These subdomain optimality conditions are coupled through Robin transmission conditions for the states and the adjoints. The parameters in the Robin transmission condition depend on the advection. This decomposition leads to a Schur complement system in which the unknowns are the state and adjoint variables on the subdomain interfaces. The Schur complement operator is the sum of subdomain Schur complement operators, the application of which is shown to correspond to the solution of subdomain optimal control problems, which are essentially smaller copies of the original optimal control problem. We show that, under suitable conditions, the application of the inverse of the subdomain Schur complement operators requires the solution of a subdomain elliptic linear-quadratic optimal control problem with Robin boundary conditions for the state.

Numerical tests for problems with distributed and with boundary control show that the dependence of the preconditioners on mesh size and subdomain size is comparable to its counterpart applied to a single advection dominated equation. These tests also show that the preconditioners are insensitive to the size of the control regularization parameter.

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1. Introduction

Optimization problems governed by (systems of) advection dominated elliptic partial differential equations (PDEs) arise in many science and engineering applications, see, e.g., [1–11], either directly or as subproblems in Newton-type or sequential quadratic optimization algorithms for the solution of optimization problems governed by (systems of) nonlinear PDEs.

* Corresponding author. Tel.: +1 713 348 5176; fax: +1 713 348 5318.

E-mail addresses: rabartl@sandia.gov (R.A. Bartlett), heinken@rice.edu (M. Heinkenschloss), dridzal@caam.rice.edu (D. Ridzal), bartv@sandia.gov (B.G. van Bloemen Waanders).

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This paper is concerned with optimization–level domain decomposition preconditioners for such problems. We focus our presentation on the linear quadratic optimal control problem

$$\text{minimize} \quad \frac{1}{2}\int_\Omega (y(x)-\hat{y}(x))^2\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega u^2(x)\,\mathrm{d}x \tag{1}$$

$$\text{subject to} \quad -\epsilon\Delta y(x) + \mathbf{a}(x)\cdot\nabla y(x) + r(x)y(x) = f(x)+u(x), \quad x\in\Omega, \tag{2a}$$

$$y(x) = 0, \quad x\in\partial\Omega_D, \tag{2b}$$

$$\epsilon\frac{\partial}{\partial\mathbf{n}}y(x) = g(x), \quad x\in\partial\Omega_N, \tag{2c}$$

where $\partial\Omega_D$, $\partial\Omega_N$ are boundary segments with $\partial\Omega_D = \partial\Omega\backslash\partial\Omega_N$, $\mathbf{a}, f, g, r, \hat{y}$ are given functions, ϵ, $\alpha > 0$ are given scalars, and $\mathbf{n}$ denotes the outward unit normal. Assumptions on these data that ensure the well-posedness of the problem will be given in the next section. The material presented in this paper can be extended to boundary control problems and several other objective functionals. The problem (1) and (2) is an optimization problem in the unknowns y and u, referred to as the state and the control, respectively.

Our domain decomposition method for the solution of (1) and (2) generalizes the Neumann–Neumann domain decomposition method, which is well known for the solution of single PDEs (see, e.g., the books [12–14]) to the optimization context. Optimization–level Neumann–Neumann domain decomposition methods for elliptic optimal control problems were first introduced in [15,16] for problems without advection. However, the presence of strong advection can significantly alter the behavior of solution algorithms and typically requires their modification. For domain decomposition methods applied to single advection dominated PDEs a nice overview of this issue is given in [14, Section 11.5.1]. The aim of our paper is to tackle this issue for optimal control problems.

The domain decomposition method presented in this paper is formulated at the optimization level. The domain Ω is partitioned into non-overlapping subdomains. Our domain decomposition methods decompose the optimality conditions for (1) and (2). Auxiliary state and so-called adjoints (Lagrange multipliers) are introduced at the subdomain interfaces. The states, adjoints, and controls in the interior of the subdomains are then viewed as implicit functions of the states and adjoints on the interface, defined through the solution of subdomain optimality conditions. To obtain a solution of the original problem (1) and (2), the states and adjoints on the interface have to be chosen such that the implicitly defined states, adjoints, and controls in the interior of the subdomains satisfy certain Robin transmission conditions at the interface boundaries. These transmission conditions take into account the advection dominated nature of the state equation and are motivated by [17,18].

The optimization-level domain decomposition described in the previous paragraph leads to a Schur complement formulation for the optimality system. The application of the Schur complement to a given vector of states and adjoints on the interface, requires the parallel solution of subdomain optimal control problems that are essentially copies of (1) and (2) restricted to the subdomains, but with Dirichlet boundary conditions at the subdomain interfaces. The Schur complement is the sum of subdomain Schur complements. Each subdomain Schur complement is shown to be invertible. The application of the inverse of each subdomain Schur complement requires the solution of another subdomain optimal control problem that is also essentially a copy of (1) and (2) restricted to the respective subdomain, but with Robin boundary conditions at the subdomain interfaces. The inverses of the subdomain Schur complements are used to derive preconditioners for the Schur complement.

Section 2 briefly reviews results on the existence, uniqueness and characterization of solutions of (1) and (2). The domain decomposition, interface conditions, subdomain Schur complements and their inverses are discussed in Section 3 using a variational point of view. The corresponding algebraic form, properties of the subdomain Schur complement matrices and some implementation details are presented in Section 4. The performance of the preconditioners on some model problems with distributed control and boundary control are documented in Section 5.

Throughout this paper we use the following notation for norms and inner products. Let $G\subset\Omega\subset\mathbb{R}^d$ or $G\subset\partial\Omega$. We define $\langle f,g\rangle_G = \int_G f(x)g(x)\,\mathrm{d}x$, $\|v\|^2_{0,G} = \int_G v^2(x)\,\mathrm{d}x$, $|v|^2_{1,G} = \int_G \nabla v(x)\cdot\nabla v(x)\,\mathrm{d}x$, and $\|v\|^2_{1,G} = \|v\|^2_{0,G} + |v|^2_{1,G}$. If $G=\Omega$ we omit G and simply write $\langle f,g\rangle$, etc.

2. The model problem

Multiplication of the advection–diffusion equation (2) by a test function

$$\phi \in Y \stackrel{\text{def}}{=} \{\phi\in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega_D\},$$

integration over Ω, and performing integration by parts leads to the following weak form:

$$a(y,\phi) + b(u,\phi) = \langle f,\phi\rangle + \langle g,\phi\rangle_{\partial\Omega_N} \quad \forall\phi\in Y, \tag{3}$$