Acoust. Sci. & Tech. **37**, 6 (2016)

ACOUSTICAL LETTER

New unconditionally stable scheme for solving two-dimensional acoustic wave problems

Di Zhang* and Xiaoping Miao†

College of Defense Engineering, PLA University of Science and Technology, NO. 1 Haifu Lane, Guanghua Road, Qinhuai District, Nanjing, 210007, China

(*Received 9 February 2016, Accepted for publication 22 May 2016*)

Keywords: Acoustic wave equation, Unconditionally stable, Finite difference scheme, Laguerre
PACS number: 43.60.Gk [doi:10.1250/ast.37.315]

1. Introduction

In this paper, we consider the following two-dimensional acoustic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = v^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + f(x,y,t) \quad (x,y) \in \Omega \times [0,T] \\ u(x,y,0) = 0, \quad \dfrac{\partial u(x,y,0)}{\partial t} = 0. \end{cases} \tag{1}$$

where u is the wave displacement, v is the sound velocity, and f is an external source. A Mur's first-order absorbing boundary condition (ABC) is considered.

$$\left[\frac{\partial u}{\partial x} - \frac{1}{c}\frac{\partial u}{\partial t}\right]_{x\in\partial\Omega} = \left[\frac{\partial u}{\partial y} - \frac{1}{c}\frac{\partial u}{\partial t}\right]_{y\in\partial\Omega} = 0 \tag{2}$$

The conventional finite difference scheme (FDS) works efficiently for solving the acoustic wave problems [1–3]. However, the FDS is an explicit time-marching algorithm, which means its time step should be limited by the Courant-Friedrichs-Levy (CFL) condition. Since time step size strongly depends on the smallest cell in a computational area, the FDS costs much time to solve acoustic wave problems with fine structures. Hence, to overcome the conflict in the calculative stability of the FDS, some unconditionally stable schemes [4–6] containing the alternating direction implicit (ADI) difference scheme have been proposed. Huang *et al.* [7] and Fu *et al.* [8] proposed a new orthogonal decomposition scheme using the associated Hermite polynomials to eliminate the CFL stability condition (AH-FDS). However, the AH-FDS has the drawback of a need for large internal storage and heavy computation.

In this work, we proposed a new unconditionally stable scheme for the acoustic wave equation using the weighted Laguerre polynomials and finite difference scheme (WLP-FDS). First, the time derivatives in the wave equation are expanded by the Laguerre polynomials and weighting functions. Since these orthogonal polynomials converge to zero with time, the sound field expanded by the weighted Laguerre polynomials converges to zero simultaneously. Then, by applying Galerkin's method and using the orthogonal property of weighted Laguerre basis functions, we can eliminate the time variables and thus obtain an unconditionally stable scheme from the computations. Finally, we can solve the implicit equation recursively and reconstruct numerical results using the expansion coefficients.

2. Formulations and WLP-FDS

Consider the Laguerre polynomials defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \quad n \geq 0; \quad t \geq 0. \tag{3}$$

The Laguerre polynomials have a recursive and simple relationship given by

$$\begin{aligned} &L_0(t) = 1 \\ &L_1(t) = 1 - t \\ &nL_n(t) = (2n - 1 - t)L_{n-1}(t) - (n-1)L_{n-2}(t) \\ &\qquad n \geq 2; \quad t \geq 0. \end{aligned} \tag{4}$$

These polynomials are orthogonal with respect to the weighting function e^{-t}.

$$\int_0^\infty L_n(t)L_p(t)e^{-t}dt = \begin{cases} 1, & n = p \\ 0, & n \neq p \end{cases} \tag{5}$$

Then, a set of orthogonal functions $\{\phi_0, \phi_1, \phi_2, \cdots\}$ can be obtained by

$$\phi_n(s \cdot t) = e^{-s\cdot t/2}L_n(s \cdot t), \tag{6}$$

where $s > 0$ is a time scale factor. Note that the functions are convergent to zero absolutely as $t \to \infty$. Then, the arbitrary functions spanned by the above functions are convergent to zero absolutely as $t \to \infty$. The above basis functions are orthogonal with respect to the scaled time $\bar{t}$ as

$$\int_0^\infty \phi_n(\bar{t}) \cdot \phi_p(\bar{t})d\bar{t} = \begin{cases} 1, & n = p \\ 0, & n \neq p, \end{cases} \tag{7}$$

where $\bar{t} = st$ is a scaled time variable. We introduce an appropriate scale factor to use the basis functions properly. The partial differential with respect to x and y can be obtained by using the above basis functions.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty}\frac{\partial}{\partial x^2}u_n(x,y)\phi_n(\bar{t}), \quad \frac{\partial^2 u}{\partial y^2} = \sum_{n=0}^{\infty}\frac{\partial}{\partial y^2}u_n(x,y)\phi_n(\bar{t}) \tag{8}$$

From [9], we obtain the first derivative of $u(x,y,t)$ versus time t as

*e-mail: zhangdirm@163.com
†e-mail: mxp57nj@163.com

$$\frac{\partial u(x,y,t)}{\partial t} = s\sum_{n=0}^{\infty}\left[0.5u_n(x,y) + \sum_{k=0,\ n>0}^{n-1} u_k(x,y)\right]\phi_n(\bar{t}). \quad (9)$$

From Eq. (9), we can conclude that the second derivative of $u(x,y,t)$ with respect to time t is

$$\frac{\partial^2 u}{\partial t^2} = s^2\sum_{n=0}^{\infty}\left[0.25u_n(x,y) + \sum_{k=0,\ n>0}^{n-1}(n-k)u_k(x,y)\right]\phi_n(\bar{t}) \quad (10)$$

Inserting Eqs. (8) and (10) into the acoustic wave equation, we can obtain

$$\begin{aligned} s^2\sum_{n=0}^{\infty}&\left[0.25u_n(x,y) + \sum_{k=0,\ n>0}^{n-1}(n-k)u_k(x,y)\right]\phi_n(\bar{t}) \\ &= v^2\sum_{n=0}^{\infty}\frac{\partial}{\partial x^2}u_n(x,y)\phi_n(\bar{t}) + v^2\sum_{n=0}^{\infty}\frac{\partial}{\partial y^2}u_n(x,y)\phi_n(\bar{t}) \quad (11) \\ &\quad + \sum_{n=0}^{\infty} f_n(x,y)\phi_n(\bar{t}). \end{aligned}$$

To eliminate the time terms $\phi_n(\bar{t})$, Galerkin's method is applied to the above equation using the orthogonal property of weighted Laguerre basis functions. We multiply both sides of Eq. (11) by $\phi_p(\bar{t})$, and then, by integrating over time from 0 to ∞, we can obtain the following equation using Eq. (7):

$$\begin{aligned} s^2&\left[0.25u_p(x,y) + \sum_{k=0,\ p>0}^{p-1}(p-k)u_k(x,y)\right] \\ &= v^2\frac{\partial^2 u_p(x,y)}{\partial x^2} + v^2\frac{\partial^2 u_p(x,y)}{\partial y^2} + f_p(x,y) \quad (12) \end{aligned}$$

where

$$f_p(x,y) = \int_0^{T_\mathrm{f}} f(x,y,t)\phi_p(\bar{t})d\bar{t}. \quad (13)$$

In Eq. (13), T_f is a finite time interval. Rewriting Eq. (12) using the finite difference scheme in space, we obtain

$$\begin{aligned} u_p(x,y)|_{i,j} &= \frac{4}{s^2}f_p(x,y)|_{i,j} - 4\sum_{k=0,\ p>0}^{p-1}(p-k)u_k(x,y)|_{i,j} \\ &+ \frac{4v^2}{s^2\cdot\Delta y_j^2}[u_p(x,y)|_{i,j+1} - 2u_p(x,y)|_{i,j} + u_p(x,y)|_{i,j-1}] \quad (14) \\ &+ \frac{4v^2}{s^2\cdot\Delta x_i^2}[u_p(x,y)|_{i+1,j} - 2u_p(x,y)|_{i,j} + u_p(x,y)|_{i-1,j}], \end{aligned}$$

where Δx_i and Δy_j denote the spatial size in the x- and y-directions, respectively. From Eq. (14), we can see that each order of a field variable is related to the adjacent four field variables. Rewriting Eq. (14), we have

$$\begin{aligned} &-\frac{4v^2}{s^2\cdot\Delta x_i^2}u_p(x,y)|_{i-1,j} - \frac{4v^2}{s^2\cdot\Delta y_j^2}u_p(x,y)|_{i,j-1} \\ &+ \left(1 + \frac{8v^2}{s^2\cdot\Delta x_i^2} + \frac{8v^2}{s^2\cdot\Delta y_j^2}\right)u_p(x,y)|_{i,j} \quad (15) \\ &- \frac{4v^2}{s^2\cdot\Delta y_j^2}u_p(x,y)|_{i,j+1} - \frac{4v^2}{s^2\cdot\Delta x_i^2}u_p(x,y)|_{i+1,j} \\ &= \frac{4}{s^2}f_p(x,y)|_{i,j} - 4\sum_{k=0,\ p>0}^{p-1}(p-k)u_k(x,y)|_{i,j}. \end{aligned}$$

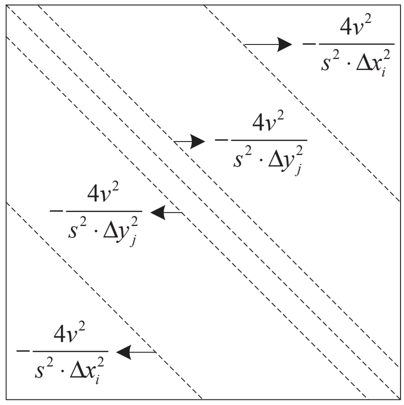

Fig. 1 Five-diagonal sparse matrix $[A]$.

Rewriting Eq. (15) in a matrix form, we can obtain

$$[A]\{u^p\} = \{f^p\} - \{(p-k)\beta^{p-1}\}. \quad (16)$$

In Eq. (16), $[A]$ is a five-diagonal matrix and its shape is shown in Fig. 1. In $[A]$, the values in central line are the coefficients of $u_p(x,y)|_{i,j}$ in Eq. (15). $\{f^p\}$ is the term of the external source due to Eq. (13). $\{\beta^{p-1}\}$ is the accumulation term from order zero to order $p-1$.

For the boundary conditions, inserting Eq. (9) into Eq. (2), the time derivative can be eliminated using Eq. (7). At $x = X$, we can obtain

$$\frac{\partial}{\partial x}u_p(x,y) - \frac{s}{c}\left[\frac{u_p(x,y)}{2} + \sum_{k=0}^{p-1}u_k(x,y)\right] = 0. \quad (17)$$

Using the central difference scheme and the averaging technique, we can obtain

$$\begin{aligned} u_p(x,y)|_{N_x-1/2,j} &= \frac{u_p(x,y)|_{N_x,j} + u_p(x,y)|_{N_x-1,j}}{2} \\ \frac{\partial}{\partial x}u_p(x,y)|_{N_x-1/2,j} &= \frac{u_p(x,y)|_{N_x,j} - u_p(x,y)|_{N_x-1,j}}{\Delta x}. \end{aligned} \quad (18)$$

Combining Eqs. (17) and (18), we have

$$\begin{aligned} &\left(-\frac{s}{4c} + \frac{1}{\Delta x}\right)u_p(x,y)|_{N_x,j} - \left(\frac{s}{4c} + \frac{1}{\Delta x}\right)u_p(x,y)|_{N_x-1,j} \\ &= \frac{s}{2c}\sum_{k=0}^{p-1}[u_p(x,y)|_{N_x,j} + u_p(x,y)|_{N_x-1,j}]. \quad (19) \end{aligned}$$

Similarly, we can obtain the ABC difference equation at $x = 0$, $y = 0$, and $y = Y$ as follows:

$$\begin{aligned} x = 0\colon &\left(-\frac{s}{4c} + \frac{1}{\Delta x}\right)u_p(x,y)|_{1,j} - \left(\frac{s}{4c} + \frac{1}{\Delta x}\right)u_p(x,y)|_{2,j} \\ &= \frac{s}{2c}\sum_{k=0}^{p-1}[u_p(x,y)|_{1,j} + u_p(x,y)|_{2,j}] \end{aligned} \quad (20)$$

$$y = 0\colon \left(-\frac{s}{4c} + \frac{1}{\Delta y}\right)u_p(x,y)|_{i,1} - \left(\frac{s}{4c} + \frac{1}{\Delta y}\right)u_p(x,y)|_{i,2}$$