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Comput. Methods Appl. Mech. Engrg. 197 (2007) 1–21

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

A discontinuous Galerkin formulation for classical and gradient plasticity. Part 2: Algorithms and numerical analysis

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Received 22 December 2006; accepted 27 June 2007
Available online 9 August 2007

Abstract

This work is the second of a two-part investigation into the use of discontinuous Galerkin methods for obtaining approximate solutions to problems of classical and gradient plasticity. Part I [J.K. Djoko, F. Ebobisse, A.T. McBride, B.D. Reddy, A discontinuous Galerkin formulation for classical and gradient plasticity. Part 1: Formulation and analysis, Comput. Methods Appl. Mech. Engrg., 196 (2007) 3881–3897] presented the formulation and analysis of such problems. This part focusses on algorithmic and computational aspects of the problem. In particular, it is shown that the predictor–corrector algorithms of classical plasticity are readily extended to the case of gradient plasticity, and to discontinuous Galerkin formulations. Conditions for convergence of the algorithms are presented, for the elastic, secant, and consistent tangent predictors. The form of the consistent tangent modulus is established for the case of gradient plasticity. A selection of numerical examples is presented and discussed with a view to illustrating aspects of the approximation scheme and algorithms, as well as features of the model of gradient plasticity adopted here.

Keywords: Discontinuous Galerkin; Elastoplasticity; Gradient plasticity; Algorithms; Consistent tangent modulus

1. Introduction

This work constitutes the second in a two-part contribution which addresses a range of issues arising in the numerical analysis and computation of problems in gradient plasticity. In [1] (hereafter referred to as Part 1), the well-posedness and convergence of semi- and fully -discrete approximations have been established for a model of gradient plasticity in which the hardening term in the yield function includes both the equivalent plastic strain as a hardening parameter, as well as its Laplacian (see Part 1 for a discussion of the relevant literature). Spatial approximations are carried out using a discontinuous Galerkin (DG) formulation in which, most generally, both the displacement and hardening parameter are approximated by piecewise-discontinuous finite element basis functions.

Various approaches have hitherto been used in numerical treatments of problems in gradient plasticity. Lasry and Belytschko [2] used a C^1 finite element formulation in a gradient theory for one-dimensional rod and spherically symmetric problems, the gradient term serving to regularise the problem and in so doing to overcome problems associated with softening. De Borst and Mühlhaus [3] derived a weak form of the gradient plasticity formulation proposed by Mühlhaus and Aifantis [4] as well as the resulting finite element framework, and used C^1 continuous interpolants of the hardening parameter to accommodate the higher-order gradient terms. Pamin and de Borst [5,6] solved the gradient enhanced problem proposed in [3] by using both C^1 and C^0 elements with a penalty constraint. Related work, also using a conforming approximation, is that of Liebe and Steinmann [7]. De Borst et al. [8] extended their earlier work to include gradient damage within a gradient

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0045-7825/$ - see front matter
doi:10.1016/j.cma.2007.06.027

plasticity formulation. Other contributions concerned with gradient damage include the investigation by Wells et al., [9] while Garikipati et al. [10] have explored a variational multiscale approach to a model of gradient plasticity proposed by Fleck and Hutchinson [11].

Various researchers have considered the extension of the gradient plasticity model in [4] to finite strains (see, for example, [12–14]). Other key contributions to the numerical simulation of problems of gradient plasticity include those of [15–22], amongst others.

As Part 1 makes clear, the goal of the present study is that of developing, analysing and implementing a DG approach to the solution of a class of plane problems in gradient plasticity. To begin with, it is essential that the appropriate variational formulation be constructed, and that conditions for existence and uniqueness of solutions be established. This has been carried out in Part 1. Furthermore, in that work it is shown that DG approximations are stable and convergent even in softening situations, for sufficiently large values of the constant in the term involving the Laplacian in the yield function.

In this work, key questions pertaining to the numerical implementation of the discrete model analysed in Part 1 are addressed. After a summary of the relevant equations is presented in Section 2, the predictor–corrector solution algorithm for this class of problems is formulated, and conditions for the convergence of the algorithm given, in Section 3. The framework in which the algorithm is formulated is that corresponding to the primal problem, which has been analysed in Part 1, and which is characterised by the flow law being written in terms of the dissipation function. The algorithms proposed here are direct extensions of those pioneered by Martin and developed in a series of papers [23–26]. The equivalence between these algorithms and those based on the flow law written as a normality condition has been established in [27,28]; in particular, the various predictors are recovered by an appropriate quadratic approximation of the non-differentiable but positively homogeneous dissipation function. A detailed treatment of these algorithmic considerations, together with proofs of convergence, are given in [29].

Conditions for the convergence of the algorithms are established first in the general context of an abstract problem, and then, as particular cases, for the elastic, secant and consistent tangent predictors. In the last case it is known [29] that it is not possible to establish conditions for unconditional convergence; this problem is overcome by introducing a perturbation of the approximation to the tangent involving a positive multiple of the identity.

Section 4 is taken up with implementational issues, and in particular with the construction of the algorithmic consistent tangent modulus. The approach taken in its derivation borrows from that for the classical theory (see for example [30]), though the derivation is more complex given the non-local nature of the problem for the hardening parameter. In addition to the tangent modulus, full details are given of the implementational aspects of the algorithm.

In Section 5 the features and performance of the algorithm are illustrated through a number of numerical examples. These address issues such as the role of softening and size dependence, and the performance of the algorithm using different moduli in the predictor step.

2. Governing equations

We are concerned with the behaviour of a body that occupies a bounded Lipschitz domain Ω in $\mathbb{R}^2$, and which undergoes small deformations. Quasi-static behaviour is assumed, so that the equation governing motion of the body is the equation of equilibrium. Elastic behaviour is specified through Hooke's law. The von Mises yield condition with linear isotropic hardening or softening is assumed to be valid, so that the space of admissible stresses $\boldsymbol{\sigma}$ and conjugate generalised stresses g is given by

$$\varphi(\boldsymbol{\sigma}, g) = |\boldsymbol{s}| + g - \kappa \leqslant 0. \tag{1}$$

Here κ is a constant related to the initial yield stress in uniaxial tension, $\boldsymbol{s}$ is the stress deviator, and g is given in terms of the scalar hardening parameter γ by

$$g(\gamma) = \begin{cases} -k_2\gamma & \text{for classical plasticity,} \\ -k_2\gamma + k_3\nabla^2\gamma & \text{for gradient plasticity,} \end{cases} \tag{2}$$

where $k_2 > 0$ defines isotropic hardening, $k_2 = 0$ for perfect plasticity, and $k_2 < 0$ corresponds to softening behaviour. The flow law for the plastic strain rate $\dot{\boldsymbol{p}}$ and hardening parameter rate $\dot{\gamma}$ then takes the alternative forms

$$\begin{aligned} &\dot{\boldsymbol{p}} = \Lambda \frac{\partial\varphi}{\partial\boldsymbol{\sigma}}, \\ &\dot{\gamma} = \Lambda, \\ &\Lambda \geqslant 0, \quad \varphi \leqslant 0, \quad \Lambda\varphi = 0, \end{aligned} \tag{3}$$

or

$$D(\boldsymbol{q}, \eta) \geqslant D(\dot{\boldsymbol{p}}, \dot{\gamma}) + \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{p}) : (\boldsymbol{q} - \dot{\boldsymbol{p}}) + g(\gamma)(\eta - \dot{\gamma}) \tag{4}$$

for arbitrary plastic strains $\boldsymbol{q}$ and hardening variables η, the stress being given by

$$\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{p}) = \mathscr{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}), \tag{5}$$

where $\mathscr{C}$ is the elasticity tensor and $\boldsymbol{\epsilon}(\boldsymbol{u})$ is the infinitesimal strain. We will make use of the latter version, known as the primal form of the flow law [29], in which the dissipation function D is given by

$$D(\boldsymbol{q}, \eta) = \begin{cases} \kappa|\boldsymbol{q}| & \text{if } |\boldsymbol{q}| \leqslant \eta, \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

We note from (3) that

$$\Lambda = \dot{\gamma} = |\dot{\boldsymbol{p}}|. \tag{7}$$

The boundary conditions are assumed for simplicity to be given by

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{and} \quad \gamma = 0 \quad \text{on } \partial\Omega, \tag{8}$$

the second condition being required only for the case of gradient plasticity. Furthermore, the initial conditions are