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Variational multiscale residual-based turbulence modeling for large eddy simulation of incompressible flows

Y. Bazilevs [a,*], V.M. Calo [a], J.A. Cottrell [a], T.J.R. Hughes [a,1], A. Reali [b], G. Scovazzi [c,2]

[a] *Institute for Computational Engineering and Sciences, The University of Texas at Austin, 201 East 24th Street, 1 University Station C0200, Austin, TX 78712, USA*
[b] *Structural Mechanics Department, University of Pavia, via Ferrata 1, 27100 Pavia, Italy*
[c] *1431 Computational Shock and Multi-physics Department, Sandia National Laboratories, P.O. Box 5800, MS 1319, Albuquerque, NM 87185-1319, USA*

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Abstract

We present an LES-type variational multiscale theory of turbulence. Our approach derives completely from the incompressible Navier–Stokes equations and does not employ any *ad hoc* devices, such as eddy viscosities. We tested the formulation on forced homogeneous isotropic turbulence and turbulent channel flows. In the calculations, we employed linear, quadratic and cubic NURBS. A dispersion analysis of simple model problems revealed NURBS elements to be superior to classical finite elements in approximating advective and diffusive processes, which play a significant role in turbulence computations. The numerical results are very good and confirm the viability of the theoretical framework.

Keywords: Variational multiscale methods; Large eddy simulation; Turbulence modeling; Optimality; Projection; Small-scale Green's function; Perturbation series; Isogeometric analysis; NURBS; Incompressible flows; Homogeneous isotropic turbulence; Turbulent channel flows

1. Introduction

Variational multiscale concepts for Large Eddy Simulation (LES) were introduced in [33]. The basic idea was to use variational projections in place of the traditional filtered equations and to focus modeling on the fine-scale equations. Avoidance of filters eliminates many difficulties associated with the traditional approach, namely, inhomogeneous non-commutative filters necessary for wall-bounded flows, use of complex filtered quantities in compressible flows, etc. In addition, modeling confined to the fine-scale equations retains numerical consistency in the coarse-scale equations and thus permits full rate-of-convergence of the underlying numerical method in contrast with the usual approach, which limits convergence rate due to artificial viscosity effects in the fully resolved scales ($O(h^{4/3})$ in the case of Smagorinsky-type models). Initial versions of the variational multiscale method focused on dividing resolved scales into coarse and fine designations, and eddy viscosities, inspired by traditional models, were only included in the fine-scale equations, and acted only on the fine scales. This version was studied in [34,36,56], and found to work very well on homogeneous isotropic flows and fully-developed equilibrium and non-equilibrium turbulent channel flows. Static eddy viscosity models were employed in these studies but superior results were subsequently obtained through the use of dynamic models, as reported in [27,41]. Good numerical results were obtained with the static approach by other investigators, namely, Collis [18], Jeanmart and Winckelmans [44], and

* Corresponding author.
E-mail address: bazily@ices.utexas.edu (Y. Bazilevs).
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Ramakrishnan and Collis [59–62]. Particular mention should be made of the work of Farhat and Koobus [20], and Koobus and Farhat [48], who have implemented this procedure in an unstructured mesh, finite element/finite volume, compressible flow code, and applied it very successfully to a number of complex test cases and industrial flows. A valuable review with many references to relevant literature may be found in [22]. We believe that this initial version of the variational multiscale concept has already demonstrated its viability and practical utility and is, at the very least, competitive with traditional LES turbulence modeling approaches. For a comprehensive treatment of multiscale concepts in turbulence, see [65]. There has also been a number of contributions to the literature in which stabilized numerical methods have been used to model turbulence (see, e.g. [26]). These endeavors are somewhat different in philosophy than the present contribution.

Nevertheless, there is still significant room for improvement. The use of traditional eddy viscosities to represent fine-scale dissipation is an inefficient mechanism. Employing an eddy viscosity in the resolved fine scales to represent turbulent dissipation introduces a consistency error, which results in the resolved fine scales being sacrificed to retain full consistency in the coarse scales. (In our opinion, this is still better than the traditional approach in which consistency in all resolved scales is sacrificed to represent turbulent dissipation.) This procedure is felt to be inefficient because approximately 7/8 of the resolved scales are typically ascribed to the fine scales. Another shortcoming noted for the initial version of the variational multiscale method is too small an energy transfer to unresolved modes when the discretization is very coarse (see, e.g. [41]). This phenomenon is also noted for some traditional models, such as the dynamic Smagorinsky model, Hughes et al. [41], but, by design, is more pronounced for the multiscale version of the dynamic model. *The objectives of recent multiscale work have been to capture all scales consistently and to avoid use of eddy viscosities altogether.* This holds the promise of much more accurate and efficient LES procedures. In this work, we describe a new variational multiscale formulation, which makes considerable progress toward these goals. In what follows, all resolved scales are viewed as coarse scales, which obviates the aforementioned issue of inefficiency *ab initio*.

We begin by taking the view that the decomposition into coarse and fine scales is exact. For example, in the spectral case, the coarse-scale space consists of all Fourier modes beneath some cut-off wave number and the fine-scale space consists of all remaining Fourier modes. Consequently, the coarse-scale space has finite dimension whereas the fine-scale space is infinite dimensional. The derivation of the coarse- and fine-scale equations proceeds, first, by substituting the split of the exact solution into coarse and fine scales into the Navier–Stokes equations, then, second, by projecting this equation into the coarse- and fine-scale subspaces. The projection into coarse scales is a finite-dimensional system for the coarse-scale component of the solution, which depends parametrically on the fine-scale component. In the spectral case, in addition to the usual terms involving the coarse-scale component, only the cross-stress and Reynolds-stress terms involve the fine-scale component. In the case of non-orthogonal bases, even the linear terms give rise to coupling between coarse and fine scales. The coarse-scale component plays an analogous role to the filtered field in the classical approach, but has the advantage of avoiding all problems associated with homogeneity, commutativity, walls, compressibility, etc. The projection into fine scales is an infinite-dimensional system for the fine-scale component of the solution, which depends parametrically on the coarse-scale component. We also assume the cut-off wave number is sufficiently large that the philosophy of LES is appropriate. For example, if there is a well-defined inertial sub-range, then we assume the cut-off wave number resides somewhere within it. This assumption enables us to further assume that the energy content in the fine scales is small compared with the coarse scales. This turns out to be important in our efforts to analytically represent the solution of the fine-scale equations. The strategy is to obtain approximate analytical expressions for the fine scales then substitute them into the coarse-scale equations which are, in turn, solved numerically. If the scale decomposition is performed in space and time, the *only* approximation in the procedure is the representation of the fine-scale solution. To provide a framework for the fine-scale approximation, we assume an infinite perturbation series expansion to treat the fine-scale nonlinear term in the fine-scale equation. By virtue of the smallness of the fine scales, this expansion is expected to converge rapidly under the circumstances described in many cases of practical interest. The remaining part of the fine-scale Navier–Stokes system is the *linearized* operator which is formally inverted through the use of a matrix Green's function. The combination of a perturbation series and Green's function provides an exact formal solution of the fine-scale Navier–Stokes equations. The driving force in these equations is the Navier–Stokes system residual computed from the coarse scales. This expresses the intuitively obvious fact that if the coarse scales constitute a good approximation to the solution of the problem, the coarse-scale residual will be small and the resulting fine-scale solution will be small as well. This is the case we have in mind and it provides a rational basis for assuming the perturbation series converges rapidly. Note that one cannot use such an argument on the original problem because in this case the perturbation series would almost definitely fail to converge. (If we could have used this argument, we would have solved the Navier–Stokes equations analytically! Unfortunately, it does not work.) The formal solution of the fine-scale equations suggests various approximations may be employed in practical problem solving. We are tempted to use the word "modeling" because approximate analytical representations of the fine scales constitute the only approximation and hence may be thought of as the "modeling" component of the present approach but we want to emphasize