Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 3493–3507
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Analysis of the equilibrated residual method for a posteriori error estimation on meshes with hanging nodes

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Received 1 November 2005; accepted 30 October 2006
Available online 15 March 2007

Dedicated to Professor Ivo Babuska on the occasion of his 80th birthday.

Abstract

The equilibrated residual method is now accepted as the best residual type a posteriori error estimator. Nevertheless, there remains a gap in the theory and practice of the method. The present work tackles the problem of existence, construction and stability of equilibrated fluxes for *hp*-finite element approximation on hybrid meshes consisting of quadrilateral and triangular elements, with hanging nodes. A practical algorithm for post-processing the finite element approximation is presented and shown to produce equilibrated fluxes for a general, one-irregular partition. The resulting fluxes are shown to be stable in the sense that the associated error estimator provides a lower bound on the local error which does not degenerate with the mesh-size. Numerical examples are included to illustrate the theoretical results.

MSC: Primary 75V05; Secondary 76M10; 65N30

Keywords: Finite element analysis; A posteriori error estimate; Equilibrated fluxes; Adaptive mesh refinement; Hanging nodes

1. Introduction

The equilibrated residual method for a posteriori error estimation is based on the solution of a local error residual problem with equilibrated boundary fluxes. The construction of equilibrated fluxes for piecewise linear finite element approximation on triangular elements goes back to the work of Kelly [15], and Ladeveze and Leguillon [16]. These works used the fluxes in conjunction with the solution of a local complementary energy variational principle to obtain error bounds. Bank and Weiser [12], again working with piecewise linear finite elements, used equilibrated fluxes to formulate a local residual boundary value problem based on the primal problem, and on the basis of numerical evidence conjectured that the resulting estimator always delivers an upper bound. This conjecture was proved in Ainsworth and Oden [5], who also extended the results to general *hp*-finite element approximation [4] and to indefinite or non-symmetric problems [6].

Despite the significant research effort directed towards equilibrated residual methods, there remain a number of quite serious gaps in the literature. For instance, it was only relatively recently shown [1] for the case of finite element approximation on triangular elements, that the equilibrated fluxes and the associated estimator are stable. Earlier work [5] either ignores this issue or relies on a saturation assumption that is difficult to verify in practical computations. There are advantages for local refinement if *hanging nodes* are allowed. The existence and stability of equilibrated fluxes for *hp*-finite element approximation on

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0045-7825/$ - see front matter
doi:10.1016/j.cma.2006.10.020

meshes with hanging nodes remains completely open: Earlier work again either relied on a saturation assumption [4], or presented equilibration procedures for particular mesh configurations, as in [11,17], without mention of issues of stability or existence in general.

The purpose of the present work is to fill these gaps. The problem of existence, construction and stability of equilibrated fluxes for *hp*-finite element approximation on hybrid meshes consisting of quadrilateral and triangular elements, with hanging nodes is considered. A practical algorithm for post-processing the finite element approximation is presented and shown to produce equilibrated fluxes for a general, one-irregular partition. An important feature of the algorithm is that the treatment of the regular and hanging nodes is completely decoupled, which simplifies the practical implementation considerably. The resulting fluxes are shown to be stable in the sense that the associated element-wise error estimator provides a lower bound on the local error. A brief informal discussion of the results presented here was given in the monograph [3].

We begin by establishing our notations and the construction of the finite element spaces in the next section. The equilibrated residual method is discussed in Section 3 and the notion of equilibrated flux moments introduced. The description and analysis of the algorithm for the determination of the flux moments forms the subject of Section 4. The stability of the actual fluxes and the resulting estimator is established in Section 5. We conclude with numerical examples that illustrate the performance of the estimator.

2. Finite element approximation

For simplicity, we shall develop the ideas in the setting of the model problem: find u such that

$$-\Delta u = f \quad \text{in } \Omega; \quad u = 0 \text{ on } \Gamma_D; \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma_N, \tag{1}$$

where Ω is a plane, polygonal domain with boundary $\partial\Omega$, with $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, $\text{meas}(\Gamma_D) > 0$ and $\Gamma_N \cap \Gamma_D = \emptyset$. The data are assumed to satisfy $f \in L_2(\Omega)$ and $g \in L_2(\Gamma_N)$. The variational formulation of this problem consists of finding $u \in V$ such that

$$B(u,v) = L(v) \quad \forall v \in V, \tag{2}$$

where V is the Hilbert space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$$

with $B : V \times V \to \mathbb{R}$ and $L : V \to \mathbb{R}$ given, respectively by

$$B(u,v) = \int_\Omega \text{grad}\, u \cdot \text{grad}\, v \, \mathrm{d}\boldsymbol{x}$$

and

$$L(v) = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} g v \, \mathrm{d}s.$$

2.1. Reference elements

A finite element in the sense of Ciarlet [13] is a triple $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ consisting of a domain $\widehat{K}$, a space $\widehat{P}$ of polynomials and a unisolvent set $\widehat{\Sigma}$ of degrees of freedom defined on $\widehat{P}$.

The pth order finite elements, $p \in \mathbb{N}$, are defined on triangular and quadrilateral reference domains

$$\widehat{K} = \begin{cases} (-1,1) \times (-1,1) & \text{(quadrilateral)} \\ \{(x,y) : x \in (-1,1), y \in (-1,-x)\} & \text{(triangle)} \end{cases}$$

with the corresponding polynomial spaces defined by

$$\widehat{P} = \begin{cases} \mathbb{Q}_p = \text{span}\{x^j y^k : 0 \leqslant j,k \leqslant p\} & \text{(quadrilateral)} \\ \mathbb{P}_p = \text{span}\{x^j y^k : 0 \leqslant j,k,j+k \leqslant p\} & \text{(triangle).} \end{cases}$$

The degrees of freedom $\widehat{\Sigma}$ are identified with a (hierarchic) set of basis functions given on the quadrilateral element by

- Four vertex or nodal functions:
  $\frac{1}{4}(1-x)(1-y), \quad \frac{1}{4}(1+x)(1-y), \; \frac{1}{4}(1+x)(1+y), \; \frac{1}{4}(1-x)(1+y);$
- $p-1$ edge functions per edge: e.g. on the edge where $x=1$:
  $\frac{1}{2}(1+x)(1-y^2)y^k, \quad \text{for } 0 \leqslant k \leqslant p-2;$
- $(p-1)^2$ interior functions:
  $(1-x^2)(1-y^2)x^j y^k, \quad \text{for } 0 \leqslant j,\, k \leqslant p-2.$

The degrees of freedom on the triangular element correspond to three nodal functions, $p-1$ edge functions per edge, and $(p-1)(p-2)/2$ interior functions. In practice, one would use a slightly different basis from the one suggested above, based on primitives of Legendre polynomials [18].

2.2. Finite element meshes and spaces

The class of finite element spaces that we shall consider is motivated by the desire to allow local refinements of the mesh and/or the local polynomial order of the approximation using a feedback finite element procedure with meshes of the type considered by Babuska and Miller [9]. The Babuska–Miller framework is based on an initial partitioning of the domain into regular finite elements with subsequent finite element meshes obtained by selectively performing subdivisions of the elements into four children while maintaining the so-called one-irregular node per element edge property. We shall now describe this procedure in more detail.