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Brief paper

Conservation and decay laws in distributed coordination control systems ☆

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ABSTRACT

In this paper we discuss and discover several conservation and associated decay laws in distributed coordination control systems, in particular in formation shape control systems. Specifically, we reveal conservations of linear momentum and angular momentum for gradient-based multi-agent formation systems modelled by single integrators, and show several corresponding conservation/decay laws for double-integrator formation stabilization systems and double-integrator flocking systems, respectively. By exploiting translation and rotation symmetry properties and insights from Noether's theorem, we further establish a multi-agent version of the relation between symmetry and conservation laws for gradient-based coordination systems derived from general potential functions, from which we generalize the conservation/decay laws to more general networked coordination control systems. The results hold in ambient spaces of any dimensions, and we focus on the 2-D and 3-D cases due to their natural interpretation as positions of agents.

1. Introduction

Formation shape control for a group of autonomous agents is concerned with designing distributed control laws such that all agents are driven to reach a configuration with specified inter-agent distances or relative positions. Formation control has been a very active research topic in the field of multi-agent coordination control, motivated by its broad applications in many areas (Oh, Park, & Ahn, 2015). In the recent decade, formation control has attracted increasing research attention and numerous results on controller design, system dynamics, convergence and stability analysis for formation systems are available; see e.g. Anderson and Helmke (2014), Deghat, Anderson, and Lin (2016) Dimarogonas and Johansson (2008), Egerstedt and Hu (2001), Krick, Broucke, and Francis (2009), Olfati-Saber and Murray (2002), and Sun, Anderson, Deghat, and Ahn (2017). In this paper we focus on momentum conservation laws and decay laws of agents' motions in distributed formation shape control systems, and aim to provide more insights on invariants and intrinsic properties for a distributed formation system as a whole. Furthermore, we also aim to explore conservation laws arising from general networked or coordination control systems that can be described as gradient flows from some general potential functions.

Conserved quantities or invariants for a physical system are quantities that remain unchanged under some transformations and usually characterize fundamental properties of a system's evolution. When specializing to a formation shape control system with networked interacting agents, we consider two types of important quantities, namely, the formation system's (overall) linear momentum and angular momentum, defined as a sum of all individual agents when their motions are described by some gradient-based control laws. The conservation of linear momentum for single-integrator formation systems can be interpreted as the invariance of formation centroid, which has been repeatedly proved in several earlier papers, e.g. Garcia de Marina, Jayawardhana, and Cao (2016a), Krick et al. (2009), Oh and Ahn (2014) and Sun and Anderson et al. (2017) under different formation potential functions and control contexts. The significance of proving an invariant formation centroid lies in the analysis of formation convergence. Note that the configuration space for all the agents in formation control is usually unbounded which prevents a direct application of commonly-used analytical tools such as LaSalle's

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invariance principle (Khalil, 2002), which require a compactness condition of certain sets in the convergence analysis. A formation system with an invariant formation centroid then allows a coordinate transformation which thus enables a correct application of LaSalle's invariance principle to prove the convergence (see detailed discussions on this issue in Krick et al. (2009)).

In this paper we will provide a comprehensive study on conservation and decay laws of linear momentum and angular momentum for both single-integrator and double-integrator formation systems. We first show the conservation laws of both linear and angular momenta when the formation systems are modelled by single-integrator gradient systems. Then we go one step further to discuss formation systems modelled by double integrators. Double-integrator models have been popularly used in studying networked systems such as formation flocking systems (Deghat et al., 2016; Olfati-Saber, 2006; Sun and Anderson et al., 2017). We will consider in detail two types of double-integrator formation systems, namely, double-integrator formation stabilization systems and double-integrator flocking systems, and further reveal several conservation or decay laws for their momentum quantities. These results constitute the first main contribution of this paper.

Invariants of a system may also be understood by and reflect some symmetry under group actions. We note that the symmetry issue in networked systems has been exploited in several papers, including (Nettleman & Goodwine, 2015; Vasile, Schwager, & Belta, 2016; Zhang, 2010), while they have mostly focused on the invariance of coordinate frame or system model reduction as a consequence of symmetry under different group actions. In the latter part of this paper, we will further exploit the symmetry property of a formation potential to derive conservation laws or momentum invariants, with the insights obtained from the celebrated Noether's theorem. To this end, a multi-agent version of the well-known relation between symmetry and conservation laws will be established, which allows generalizations of the conservation/decay laws from formation systems to other networked control systems. These generalizations on networked coordination systems constitute the second main contribution of this paper. We also note that conservation or decay properties of momentum quantities play a significant role in multi-agent coordination control, and typical applications include system reduction in optimal formation collision avoidance (see e.g. Hu & Sastry, 2001) and formation steering control (see e.g. Garcia de Marina et al., 2016a; Markdahl, Karayiannidis, Hu, & Kragic, 2012).

This paper is organized as follows. In Section 2, preliminary concepts on graph theory and formation systems (including both single-integrator models and double-integrator models) are introduced. In Section 3, we discuss the conservation laws for both linear momentum and angular momentum for single-integrator formation systems. Sections 4 and 5 focus on conservation/decay laws of momentum quantities for double-integrator formation stabilization system and for double-integrator flocking system, respectively. Discussions with new insights from Noether's theorem, as well as generalizations and applications of the conservation/decay laws, are provided in Section 6. Finally, Section 7 concludes this paper.

2. Preliminaries and formation system equations

2.1. Preliminaries

Consider an undirected simple graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbour set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. We define an *orientated* incidence matrix $H \in \mathbb{R}^{m\times n}$ for the *undirected* graph $\mathcal{G}$ by assigning an *arbitrary* orientation for each edge. Note that for a rigid formation modelled as an *undirected* graph considered in this paper, the orientation of each edge for writing the incidence matrix can be defined arbitrarily and the stability and convergence analysis in the next sections remains unchanged. Following this, we define the entries of H as $h_{ki} = +1$ if the kth edge sinks at node i, or $h_{ki} = -1$ if the kth edge leaves node i, or $h_{ki} = 0$ otherwise. The adjacency matrix $A(\mathcal{G})$ is a symmetric $n \times n$ matrix encoding the vertex adjacency relationships, with entries $a_{ij} = 1$ if $\{i, j\} \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The Laplacian matrix $L(\mathcal{G})$ is also often used for matrix representation of a graph $\mathcal{G}$, which is defined as $L(\mathcal{G}) = H^\top H$ for undirected graphs. For a connected undirected graph, there holds $\mathrm{rank}(L) = n - 1$ and $\mathrm{null}(L) = \mathrm{null}(H) = \mathrm{span}\{\mathbf{1}_n\}$.

Let $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$ denotes a point that is assigned to agent $i \in \mathcal{V}$. The stacked vector $p = [p_1^\top, p_2^\top, \dots, p_n^\top]^\top \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework (specifically, a *formation* in the context of formation control) of $\mathcal{G}$ in $\mathbb{R}^d$. By introducing the matrix $\bar{H} := H \otimes I_{d\times d} \in \mathbb{R}^{dm\times dn}$, one can construct the relative position vector z as follows

$$z = \bar{H}p \tag{1}$$

where $z = [z_1^\top, z_2^\top, \dots, z_m^\top]^\top \in \mathbb{R}^{dm}$, with $z_k \in \mathbb{R}^d$ being the relative position vector for the vertex pair defined by the kth edge. Let $d_{k_{ij}}$ denote the desired length of edge k which links agent i and j. We further define (for an arbitrary formation)

$$e_{k_{ij}} = \|p_i - p_j\|^2 - d_{k_{ij}}^2 = \|z_{k_{ij}}\|^2 - d_{k_{ij}}^2 \tag{2}$$

to denote the squared distance error for edge k. Note we may also use e_k and d_k occasionally for notational convenience in the sequel if no confusion is expected. The squared distance error vector is denoted by $e = [e_1, e_2, \dots, e_m]^\top$.

The problem of determining conservation/decay laws studied in this paper has its origins in the problem of formation control of rigid shapes. The definition of graph rigidity can be found in e.g. Hendrickson (1992). Define $Z(z) = \mathrm{diag}(z_1, z_2, \dots, z_m) \in \mathbb{R}^{dm\times m}$. With this notation at hand, we consider the smooth distance map $r_{\mathcal{G}} : \mathbb{R}^{dn} \longrightarrow \mathbb{R}^m, r_{\mathcal{G}}(p) = (\|p_i - p_j\|^2)_{(i,j)\in\mathcal{E}} = Z^\top z$. A useful tool to study graph rigidity is the **rigidity matrix**, which is defined as the Jacobian matrix $R(p) = \frac{1}{2}\partial r_{\mathcal{G}}(p)/\partial(p) = Z(z)^\top \bar{H} \in \mathbb{R}^{m\times dn}$. We refer the readers to Hendrickson (1992) for applications of rigidity matrix on characterizing infinitesimal rigidity of a framework. We note that the results in this paper do not depend on any rigidity assumption of a target formation shape.

2.2. Motion equations: single-integrator formation system

Most papers (see e.g. Anderson & Helmke, 2014; Krick et al., 2009; Oh & Ahn, 2014) on rigid formation control have considered the following formation control system modelled by a single integrator

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i}(\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j), \quad i = 1, \dots, n \tag{3}$$

which defines the *steepest descent gradient flow* of the distance potential function

$$V(p) = \frac{1}{4}\sum_{(i,j)\in\mathcal{E}}(\|p_i - p_j\|^2 - d_{k_{ij}}^2)^2. \tag{4}$$

In a compact form, we can rewrite (3) as

$$\dot{p}(t) = -\nabla_p V = -R^\top(z)e(z) \tag{5}$$